Lecture 23

LEM 27 With above notation, set

$$\rho = \frac{1}{2} \sum_{\beta \in \Phi^+} \beta$$

Then

$$(2\rho, \alpha) = (\alpha, \alpha) \qquad \forall \alpha \in \Pi$$

pf Write

$$\rho = \frac{\alpha}{2} + \frac{1}{2} \sum_{\substack{\beta \in \Phi^+ \\ \beta \neq \alpha}} \beta$$

Apply s_α:

$$\rho \longrightarrow -\frac{\alpha}{2} + \frac{1}{2} \sum_{\substack{\beta \in \Phi^+ \\ \beta \neq \alpha}} \beta$$

$$= -\frac{\alpha}{2} + \rho - \frac{\alpha}{2}$$

$$= \rho - \alpha$$

But $s_\alpha(\rho) = \rho - \langle \alpha^\vee, \rho \rangle \alpha$

So $\langle \alpha^\vee, \rho \rangle = 1$

Recall $\langle \alpha^\vee, \rho \rangle = (\nu(\alpha^\vee), \rho)$

$$= \left(\frac{\alpha}{d_\alpha}, \rho \right)$$

and $d_\alpha = \frac{(\alpha, \alpha)}{2}$

Result follows. $\square$

DEF 28 $\quad \forall \lambda \in Q$ define

$$K_\lambda = K_{\alpha_1}^{a_1} K_{\alpha_2}^{a_2} \cdots K_{\alpha_n}^{a_n}$$

where

$$\lambda = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

LEM 29 $\quad$ The following hold in $\tilde{U}_q$

(i) $\quad K_\lambda K_\mu = K_{\lambda+\mu} \qquad \lambda, \mu \in Q$

(ii) $\quad K_0 = I$

(iii) $\quad K_\lambda^{-1} = K_{-\lambda}$

pf ROUTINE $\quad \square$

LEM 30 For $\tilde{U}_q$,

$$\text{(i)} \quad k_\lambda e_\beta k_\lambda^{-1} = q^{(\lambda,\beta)} e_\beta \qquad \lambda \in Q, \ \beta \in \Pi$$

$$\text{(ii)} \quad k_\lambda f_\beta k_\lambda^{-1} = q^{-(\lambda,\beta)} f_\beta \qquad \text{"}$$

pf (i) Use Def 28 and

$$k_\alpha e_\beta k_\alpha^{-1} = q^{(\alpha,\beta)} e_\beta \qquad \alpha, \beta \in \Pi$$

(ii) Sim $\square$

THM 31 For $\tilde{U}_q$ we have

$$S^2(x) = T^{-1} x T \qquad x \in \tilde{U}_q$$

where

$$T = K_{2\rho} \qquad \rho \text{ from Lem 27}$$

pf The maps

$$x \longrightarrow S^2(x)$$
$$x \longrightarrow T^{-1} x T$$

are both alg homs. To show they are equal, show that they agree at $x \in \{ k_\alpha, k_\alpha^{-1}, e_\alpha, f_\alpha \mid \alpha \in \Pi \}$

We have

$$S^2(k_\alpha^{\pm 1}) = k_\alpha^{\pm 1} \qquad \alpha \in \Pi$$
$$S^2(e_\alpha) = q^{-(\alpha,\alpha)} e_\alpha$$
$$S^2(f_\alpha) = q^{(\alpha,\alpha)} f_\alpha$$

Indeed for $\alpha \in \Pi$,

$$k_\alpha \xrightarrow[S]{} k_\alpha^{-1} \xrightarrow[S]{} k_\alpha$$

$$e_\alpha \xrightarrow[S]{} -k_\alpha^{-1}e_\alpha \xrightarrow[S]{} -S(e_\alpha)S(k_\alpha^{-1}) = -(-k_\alpha^{-1}e_\alpha)k_\alpha$$

$$= k_\alpha^{-1}e_\alpha k_\alpha$$

$$= q^{-(\alpha,\alpha)}e_\alpha$$

$$f_\alpha \xrightarrow[S]{} -f_\alpha k_\alpha \xrightarrow[S]{} -S(k_\alpha)S(f_\alpha) = -k_\alpha^{-1}(-f_\alpha k_\alpha)$$

$$= k_\alpha^{-1}f_\alpha k_\alpha$$

$$= q^{(\alpha,\alpha)}f_\alpha$$

Also for $\alpha \in \Pi$,

$$T^{-1}k_\alpha^{\pm 1}T = k_\alpha$$

$$T^{-1}e_\alpha T = K_{-2\rho}\, e_\alpha\, k_{2\rho}$$

$$= q^{-(2\rho,\alpha)}e_\alpha \qquad \text{by Lem 30}$$

$$= q^{-(\alpha,\alpha)}e_\alpha \qquad \text{by Lem 27}$$

Sim

$$T^{-1}f_\alpha T = q^{(2\rho,\alpha)}f_\alpha$$

$$= q^{(\alpha,\alpha)}f_\alpha.$$

Result follows. $\square$

COR 32 For $\tilde{U}_q$,

$$S^{-1}(x) = K_{2\rho}\, S(x)\, K_{2\rho}^{-1} \qquad x \in \tilde{U}_q$$

pf By Thm 31 and

$$S(K_{2\rho}) = K_{2\rho}^{-1}$$

$\square$

Next goal: We found earlier a Hopf alg str

$$(\tilde{U}_q,\ \Delta,\ \varepsilon,\ S)$$

We next transport this structure to U_q.

To do this, we show that Δ, ε, S "respect"

the q-Serre relations

DEF 33 For distinct $\alpha, \beta \in \Pi$ define elements $u^{\pm}_{\alpha\beta} \in \tilde{U}_q$:

$$u^{+}_{\alpha\beta} = \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_{\alpha} (-1)^{s} e_{\alpha}^{1-A_{\alpha\beta}-s} e_{\beta} e_{\alpha}^{s}$$

$$u^{-}_{\alpha\beta} = \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_{\alpha} (-1)^{s} f_{\alpha}^{1-A_{\alpha\beta}-s} f_{\beta} f_{\alpha}^{s}$$

NOTE By constr we have alg hom

$$\tilde{U}_q \to U_q$$

let J = kernel.

Then J is generated (as a 2-sided ideal of $\tilde{U}_q$) by

$$u^{\pm}_{\alpha\beta}, \qquad \alpha, \beta \in \Pi, \quad \alpha \neq \beta$$

Ie

$$J = \operatorname{Span}\left\{ x\, u^{\pm}_{\alpha\beta}\, y \;\middle|\; x, y \in \tilde{U}_q, \quad \alpha, \beta \in \Pi, \; \alpha \neq \beta \right\}$$

We next compute

$$\Delta(u^{\pm}_{\alpha\beta}), \qquad \varepsilon(u^{\pm}_{\alpha\beta}), \qquad S(u^{\pm}_{\alpha\beta})$$

Start with Δ.

LEM 34 For $\alpha \in \Pi$ and $n \in \mathbb{N}$ the following holds in $\tilde{U}_q$:

(i) $$\Delta(e_\alpha^n) = \sum_{r=0}^{n} q_\alpha^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha e_\alpha^{n-r} k_\alpha^r \otimes e_\alpha^r$$

(ii) $$\Delta(f_\alpha^n) = \sum_{r=0}^{n} q_\alpha^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_\alpha f_\alpha^r \otimes f_\alpha^{n-r} k_\alpha^{-r}$$

pf Apply the alg hom

$$U_{q_\alpha}(sl_2) \longrightarrow \tilde{U}_q$$

$$e \to e_\alpha$$

$$f \to f_\alpha$$

$$k^{\pm 1} \longrightarrow k_\alpha^{\pm 1}$$

to the formula for $\Delta(e^n)$, $\Delta(f^n)$ in $U_{q_\alpha}(sl_2)$ $\square$

LEM 35 For distinct $\alpha, \beta \in \Pi$ the following hold in $\tilde{U}_q$:

(i) $$\Delta(u^+_{\alpha\beta}) = u^+_{\alpha\beta} \otimes I + K_\alpha^{1-A_{\alpha\beta}} K_\beta \otimes u^+_{\alpha\beta}$$

(ii) $$\Delta(u^-_{\alpha\beta}) = u^-_{\alpha\beta} \otimes K_\alpha^{A_{\alpha\beta}-1} K_\beta^{-1} + I \otimes u^-_{\alpha\beta}$$

pf (i) By Def 33

$$\Delta(u^+_{\alpha\beta}) = \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s \, \Delta(e_\alpha^{1-A_{\alpha\beta}-s}) \underbrace{\Delta(e_\beta)}_{\substack{\shortparallel \\ e_\beta \otimes I + k_\beta \otimes e_\beta}} \Delta(e_\alpha^s)$$

show

$$\sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s \, \Delta(e_\alpha^{1-A_{\alpha\beta}-s})(e_\beta \otimes I) \, \Delta(e_\alpha^s) = u^+_{\alpha\beta} \otimes I \qquad (*)$$

$$\sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s \, \Delta(e_\alpha^{1-A_{\alpha\beta}-s})(k_\beta \otimes e_\beta) \, \Delta(e_\alpha^s) = k_\alpha^{1-A_{\alpha\beta}} k_\beta \otimes u^+_{\alpha\beta} \qquad (**)$$

To do this, use Lem 34 (tedious but routine ex) $\square$

(ii) Similar

LEM 36 For distinct $\alpha, \beta \in \Pi$ this holds in $\widetilde{U}_q$:

(i) $\varepsilon(u^+_{\alpha\beta}) = 0$,

(ii) $\varepsilon(u^-_{\alpha\beta}) = 0$

pf (i) obs

$$\varepsilon(u^+_{\alpha\beta}) = \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s \, \varepsilon\left(e_\alpha^{1-A_{\alpha\beta}-s}\right) \underbrace{\varepsilon(e_\beta)}_{=0} \varepsilon(e_\alpha^s)$$

$$= 0$$

(ii) Sim. $\square$

LEM 37 For $\alpha \in \Pi$ and $n \in \mathbb{N}$ the following holds in $\tilde{U}_q$:

(i) $S(e_\alpha^n) = (-1)^n k_\alpha^{-n} e_\alpha^n q_\alpha^{n(n-1)}$

(ii) $S(f_\alpha^n) = (-1)^n f_\alpha^n k_\alpha^n q_\alpha^{-n(n-1)}$

pf Apply the alg hom

$$\begin{aligned} U_{q_\alpha}(sl_2) &\longrightarrow \tilde{U}_q \\ e &\to e_\alpha \\ f &\to f_\alpha \\ k^{\pm 1} &\to k_\alpha^{\pm 1} \end{aligned}$$

to the formula for

$S(e^n)$, $S(f^n)$ in $U_{q_\alpha}(sl_2)$ $\square$

LEM 38 For distinct $\alpha, \beta \in \Pi$ the following holds in $\tilde{U}_q$:

(i) $S(u^+_{\alpha\beta}) = -K_\alpha^{A_{\alpha\beta}-1} K_\beta^{-1} u^+_{\alpha\beta}$

(ii) $S(u^-_{\alpha\beta}) = -u^-_{\alpha\beta} K_\alpha^{1-A_{\alpha\beta}} K_\beta$

pf (i)

$$S(u_{\alpha\beta}^{+}) = \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s \, S\left(E_\alpha^{1-A_{\alpha\beta}-s} E_\beta E_\alpha^s\right)$$

$$= \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s \, S(E_\alpha^s)\, S(E_\beta)\, S\left(E_\alpha^{1-A_{\alpha\beta}-s}\right)$$

Now expand using LEM 37 and simplify.

details:

$$= \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s \left((-1)^s K_\alpha^{-s} E_\alpha^s q_\alpha^{s(s-1)}\right)$$

$$(-K_\beta^{-1} E_\beta)\left((-1)^{1-A_{\alpha\beta}-s} K_\alpha^{s+A_{\alpha\beta}-1} E_\alpha^{1-A_{\alpha\beta}-s} q_\alpha^{(1-A_{\alpha\beta}-s)(-A_{\alpha\beta}-s)}\right)$$

$$= \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s (-1)^s q_\alpha^{s(s-1)} (-1)(-1)^{1-A_{\alpha\beta}-s}$$

$$q_\alpha^{(1-A_{\alpha\beta}-s)(-A_{\alpha\beta}-s)}$$

$$K_\alpha^{-s} E_\alpha^s K_\beta^{-1} E_\beta K_\alpha^{s+A_{\alpha\beta}-1} E_\alpha^{1-A_{\alpha\beta}-s}$$

$$= \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha q_\alpha^{s(s-1)} (-1)^{A_{\alpha\beta}+s} q_\alpha^{(1-A_{\alpha\beta}-s)(-A_{\alpha\beta}-s)}$$

$$q_\alpha^{sA_{\alpha\beta}} q_\alpha^{-2s(s+A_{\alpha\beta}-1)} q_\alpha^{-A_{\alpha\beta}(s+A_{\alpha\beta}-1)}$$

$$K_\alpha^{A_{\alpha\beta}-1} K_\beta^{-1} E_\alpha^s E_\beta E_\alpha^{1-A_{\alpha\beta}-s}$$

$$= -K_\alpha^{A_{\alpha\beta}-1} K_\beta^{-1} \sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^{1-A_{\alpha\beta}-s} E_\alpha^s E_\beta E_\alpha^{1-A_{\alpha\beta}-s}$$

$$= -K_\alpha^{A_{\alpha\beta}-1} K_\beta^{-1} u_{\alpha\beta}^{+}$$

$\square$

Recall we have surjective alg hom

$$\tilde{U}_q \longrightarrow U_q \qquad *$$

thm 39 $\exists$ unique Hopf-alg str $(U_q, \Delta, \varepsilon, S)$ s.t $*$ is hom of Hopf algebras. The following holds in U_q

x	$\Delta(x)$	$\varepsilon(x)$	$S(x)$
K_α	$K_\alpha \otimes K_\alpha$	1	K_α^{-1}
K_α^{-1}	$K_\alpha^{-1} \otimes K_\alpha^{-1}$	1	K_α
E_α	$E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha$	0	$-K_\alpha^{-1} E_\alpha$
F_α	$F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha$	0	$-F_\alpha K_\alpha$

pf Uniqueness — this part is similar to Prop 26

Find $\Delta(K_\alpha)$. Recall this commutes:

$$\begin{tikzcd}
\tilde{U} \arrow[r] \arrow[d, "\Delta"'] & U \arrow[d, "\Delta"] \\
\tilde{U} \otimes \tilde{U} \arrow[r] & U \otimes U
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{K}_\alpha \arrow[r, mapsto] \arrow[d, mapsto] & K_\alpha \arrow[d, mapsto] \\
\tilde{K}_\alpha \otimes \tilde{K}_\alpha \arrow[r, mapsto] & K_\alpha \otimes K_\alpha
\end{tikzcd}$$

$\Delta(K_\alpha) = ?$

Must have $\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$

The table is found in this manner.

Existence

Def Δ, ε, S as in table.

show

(i) Δ is alg hom

(ii) ε is alg hom

(iii) S is alg antihom

(iv) various diagrams commute

(i) Need to check Δ respects the q-Serre rels.

For $\alpha, \beta \in \Pi$ $\alpha \neq \beta$

$$\Delta(u_{\alpha\beta}^{+}) = u_{\alpha\beta}^{+} \otimes I + K_\alpha^{1-A_{\alpha\beta}} K_\beta \otimes u_{\alpha\beta}^{+} \quad (\text{in } \tilde{U}_q)$$

$$= 0 \otimes I + K_\alpha^{1-A_{\alpha\beta}} K_\beta \otimes 0 \quad (\text{in } U_q)$$

$$= 0 \quad \text{in } (U_q)$$

Sim $\Delta(u_{\alpha\beta}^{-}) = 0$ (in U_q)

(ii) Need to check ε respects q-Serre rel

$$\varepsilon(u_{\alpha\beta}^{+}) = 0 \quad (\text{in } \tilde{U}_q)$$

$$= 0 \quad (\text{in } U_q)$$

$$(\text{in } U_q)$$

Sim $\varepsilon(u_{\alpha\beta}^{-}) = 0$

(iii) Need to check S respects q-Serre rel

$$S(u_{\alpha\beta}^{+}) = -K_\alpha^{A_{\alpha\beta}-1} K_\beta^{-1} u_{\alpha\beta}^{+} \quad (\text{in } \tilde{U}_q)$$

$$= -K_\alpha^{A_{\alpha\beta}-1} K_\beta^{-1}\, 0 \quad (\text{in } U_q)$$

$$= 0 \quad (\text{in } U_q)$$

Sim $S(u_{\alpha\beta}^{-}) = 0$ (in U_q)

(iv) The diagrams commute in U_q since they commute in $\tilde{U}_q$

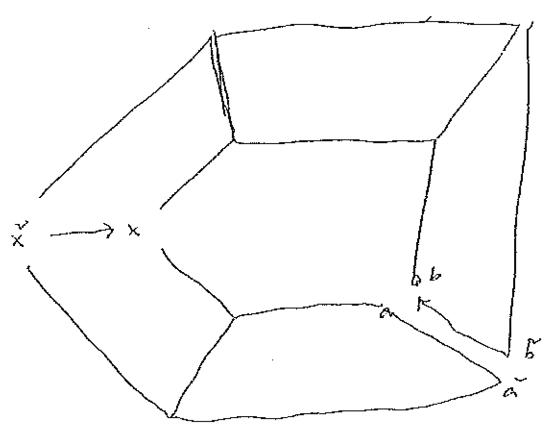
To illustrate
show $\Delta : U \to U \otimes U$ is co-assoc
By constr this diagram commutes:

$$\begin{tikzcd} \tilde{U} \arrow[r, "\Delta"] \arrow[d, "*"'] & \tilde{U}\otimes\tilde{U} \arrow[d, "*\otimes *"] \\ U \arrow[r, "\Delta"'] & U \otimes U \end{tikzcd}$$

Consider

$$\begin{tikzcd} & & \tilde{U}\otimes\tilde{U} \arrow[rr] \arrow[d, "*\otimes *"] & & (\tilde{U}\otimes\tilde{U})\otimes\tilde{U} \arrow[dl, dash] \arrow[ddd, "can"] \\ & & U\otimes U \arrow[r, "\Delta\otimes 1"] & (U\otimes U)\otimes U \arrow[dd, "can"] & \\ \tilde{U} \arrow[r, "*"] \arrow[uurr, "\Delta"] \arrow[ddrr, "\Delta"'] & U \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & & \\ & & U\otimes U \arrow[r, "1\otimes\Delta"'] & U\otimes(U\otimes U) \arrow[dr, dash] & \\ & & \tilde{U}\otimes\tilde{U} \arrow[u, dash] \arrow[rr] & & \tilde{U}\otimes(\tilde{U}\otimes\tilde{U}) \end{tikzcd}$$

show inner pentagon commutes.
By constr outer pentagon commutes
pick $x \in U$ let $\tilde{x} \in \tilde{U}$ denote preimage under $*$



show $a = b$

$\tilde{a} = \tilde{b}$
so $a = b$

$\square$

Other diagrams are sim checked.