Lecture 22

A grading for $\tilde{U}_q$ and U_q

Def 21 $\quad \forall \beta \in Q$ define

$$\tilde{U}_\beta = \{ x \in \tilde{U}_q \mid k_\alpha x k_\alpha^{-1} = q^{(\alpha,\beta)} x \quad \forall \alpha \in \Pi \}$$

obs

$$1 \in \tilde{U}_0$$

$$K_\alpha^{\pm} \in \tilde{U}_0 \qquad \forall \alpha \in \Pi$$

$$E_\beta \in \tilde{U}_\beta \qquad \forall \beta \in \Pi$$

$$F_\beta \in \tilde{U}_{-\beta}$$

Thm 22 The following is a grading for $\tilde{U}_q$:

$$\tilde{U}_q = \sum_{\beta \in Q} \tilde{U}_\beta \qquad (ds)$$

pf First show

$$\tilde{U}_\beta \tilde{U}_{\beta'} \subseteq \tilde{U}_{\beta+\beta'} \qquad \forall \beta, \beta' \in \Pi$$

Given $x \in \tilde{U}_\beta$ $y \in \tilde{U}_{\beta'}$

show $xy \in \tilde{U}_{\beta+\beta'}$

$\forall \alpha \in \Pi$

$$\begin{aligned} K_\alpha xy K_\alpha^{-1} &= K_\alpha x K_\alpha^{-1} K_\alpha y K_\alpha^{-1} \\ &= q^{(\alpha,\beta)} x \quad q^{(\alpha,\beta')} y \\ &= q^{(\alpha, \beta+\beta')} xy \end{aligned}$$

So $xy \in \tilde{U}_{\beta+\beta'}$ ✓

Obs $\tilde{U}_q = \sum_{\beta \in Q} \tilde{U}_\beta$

since RHS is subalg of $\tilde{U}_q$ that contains the gens of $\tilde{U}_q$

Sum is direct since the $\tilde{U}_\beta$ are common eigenspaces for the $\{K_\alpha\}_{\alpha \in \Pi}$ and assoc characters are distinct.

$\square$

We now turn to U_q

DEF 23 $\forall \beta \in \Phi$ define

$$U_\beta = \left\{ x \in U_q \;\middle|\; k_\alpha x k_\alpha^{-1} = q^{(\alpha,\beta)} x \quad \forall \alpha \in \Pi \right\}$$

THM 24 The following is a grading of U_q:

$$U_q = \sum_{\beta \in \Phi} U_\beta \qquad \text{ds.}$$

Pf Apply the hom $\tilde{U}_q \to U_q$ to the grading in Thm 22 $\square$

We will obtain info about $\tilde{U}_q$ by treating it as a module for $U_{q_\alpha}(sl_2)$ $\forall \alpha \in \Pi$.

We do this as follows:

- We view $\tilde{U}_q$ as $\tilde{U}_q$-module via adjoint action (defined below)
- Using our hom $U_{q_\alpha}(sl_2) \to \tilde{U}_q$, $\tilde{U}_q$ becomes a module for $U_{q_\alpha}(sl_2)$.

LEM 25 For a Hopf alg $(A, \Delta, \varepsilon, S)$

the vs A becomes an A-module with action

$$\begin{array}{ccccc} A & \times & A & \longrightarrow & A \\ x & & a & \longrightarrow & \sum_i x_i\, a (x_i')^S \end{array}$$

where $\Delta(x) = \sum_i x_i \otimes x_i'$

Call this the <u>quantum adjoint</u> action

pf For $a \in A$

$1 . a = a$ since $\Delta(1) = 1 \otimes 1$, $S(1) = 1$

For $x, y \in A$ check

$$x.(y.a) \stackrel{?}{=} (xy).a$$

Write $\Delta(x) = \sum_i x_i \otimes x_i'$

$\Delta(y) = \sum_j y_j \otimes y_j'$

$$\Delta(xy) = \Delta(x)\Delta(y) = \sum_i \sum_j x_i y_j \otimes x_i' y_j'$$

$$x \cdot (y \cdot a) = x \cdot \left(\sum_j y_j \, a (y_j')^S \right)$$

$$= \sum_i x_i \left(\sum_j y_j \, a (y_j')^S \right) (x_i')^S$$

$$= \sum_i \sum_j x_i y_j \, a \underbrace{(y_j')^S (x_i')^S}_{= (x_i' y_j')^S}$$

$$= (xy) \cdot a \quad \checkmark$$

$\square$

We now return to $U_q(\mathfrak{g})$, standard setup.

We next define a Hopf alg str on $\tilde{U}_q = \tilde{U}_q(\mathfrak{g})$

Recall $\forall \alpha \in \Pi$ we have an alg hom

$$\begin{aligned} U_{q_\alpha}(sl_2) &\longrightarrow \tilde{U}_q \\ k &\longrightarrow k_\alpha \\ k^{-1} &\longrightarrow k_\alpha^{-1} \\ e &\longrightarrow e_\alpha \\ f &\longrightarrow f_\alpha \end{aligned} \qquad (*)$$

THM 26 $\exists$ unique Hopf alg str $(\tilde{U}_q, \Delta, \varepsilon, S)$ st $(*)$ is a Hom of Hopf algebras $\forall \alpha \in \Pi$

the Δ, ε, S satisfy:

x	$\Delta(x)$	$\varepsilon(x)$	$S(x)$
k_α	$k_\alpha \otimes k_\alpha$	1	k_α^{-1}
k_α^{-1}	$k_\alpha^{-1} \otimes k_\alpha^{-1}$	1	k_α
e_α	$e_\alpha \otimes 1 + k_\alpha \otimes e_\alpha$	0	$-k_\alpha^{-1} e_\alpha$
f_α	$f_\alpha \otimes k_\alpha^{-1} + 1 \otimes f_\alpha$	0	$-f_\alpha k_\alpha$

Pf <u>Uniqueness</u>

Assume Hopf alg str $(\tilde{U}_q, \Delta, \varepsilon, S)$ exists.
show it satisfies table. Given $\alpha \in \Pi$.

<u>Find $\Delta(k_\alpha)$</u>

Since ψ is hom of Hopf algebras this diag commutes:

$$\begin{tikzcd} U_{q_\alpha}(sl_2) \arrow[r] \arrow[d, "\Delta"'] & \tilde{U}_q \arrow[d, "\Delta"] \\ U_{q_\alpha}(sl_2) \otimes U_{q_\alpha}(sl_2) \arrow[r] & \tilde{U}_q \otimes \tilde{U}_q \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r, mapsto] \arrow[d, mapsto] & k_\alpha \arrow[d, mapsto] \\ k \otimes k \arrow[r, mapsto] & \Delta(k_\alpha) \end{tikzcd}$$

$\Delta(k_\alpha)$ $\overset{\text{require}}{=}$ $k_\alpha \otimes k_\alpha$

$$\Delta(k_\alpha) = k_\alpha \otimes k_\alpha$$

$\Delta(k_\alpha^{-1})$, $\Delta(e_\alpha)$, $\Delta(f_\alpha)$ sim found.

Find $\varepsilon(k_\alpha)$

Since $*$ is Hopf alg hom this diag commutes:

$$\begin{tikzcd} U_{q_\alpha}(sl_2) \arrow[r] \arrow[d, "\varepsilon"'] & \widetilde{U}_q \arrow[d, "\varepsilon"] \\ \mathbb{F} \arrow[r, "id"'] & \mathbb{F} \end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r, mapsto] \arrow[d, mapsto] & k_\alpha \arrow[d, mapsto] \\ 1 \arrow[r, mapsto] & \varepsilon(k_\alpha) \\ & 1 \end{tikzcd}$$

" require

$\varepsilon(k_\alpha) = 1.$

$\varepsilon(k_\alpha^{-1})$, $\varepsilon(e_\alpha)$, $\varepsilon(f_\alpha)$ sim found.

Find $S(k_\alpha)$

Since $*$ is Hopf alg hom this diag commutes:

$$\begin{tikzcd}
U_{q_\alpha}(sl_2) \arrow[r] \arrow[d, "S"'] & \tilde{U}_q \arrow[d, "S"] \\
U_{q_\alpha}(sl_2) \arrow[r] & \tilde{U}_q
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[r, mapsto] \arrow[d, mapsto] & k_\alpha \arrow[d, mapsto] \\
 & S(k_\alpha) \arrow[d, equal, "\text{require}"] \\
k^{-1} \arrow[r, mapsto] & k_\alpha^{-1}
\end{tikzcd}$$

$$S(k_\alpha) = k_\alpha^{-1}$$

$S(k_\alpha^{-1})$, $S(e_\alpha)$, $S(f_\alpha)$ sim found.

We have shown the Hopf alg $(\tilde{u}_q, \Delta, \varepsilon, S)$ is unique if it exists.

<u>Existence</u> Let Δ, ε, S be as in table.

We must check

(i) Δ is an algebra hom

(ii) ε ...

(iii) S is antihom

(iv) Δ is coassociative

(v) these commute:

$$\begin{tikzcd} \tilde{U}_q \arrow[r, "\Delta"] \arrow[d, "I"'] & \tilde{U}_q \otimes \tilde{U}_q \arrow[d, "\varepsilon \otimes I"] \\ \tilde{U}_q \arrow[r, "can"'] & \mathbb{F} \otimes \tilde{U}_q \end{tikzcd} \qquad \begin{tikzcd} \tilde{U}_q \arrow[r, "\Delta"] \arrow[d, "I"'] & \tilde{U}_q \otimes \tilde{U}_q \arrow[d, "I \otimes \varepsilon"] \\ \tilde{U}_q \arrow[r, "can"'] & \tilde{U}_q \otimes \mathbb{F} \end{tikzcd}$$

(vi) these commute:

$$\begin{tikzcd} \tilde{U}_q \arrow[r, "\Delta"] \arrow[d, "i \circ \varepsilon"'] & \tilde{U}_q \otimes \tilde{U}_q \arrow[d, "S \otimes I"] \\ \tilde{U}_q & \tilde{U}_q \otimes \tilde{U}_q \arrow[l, "mult"] \end{tikzcd} \qquad \begin{tikzcd} \tilde{U}_q \arrow[r, "\Delta"] \arrow[d, "i \circ \varepsilon"'] & \tilde{U}_q \otimes \tilde{U}_q \arrow[d, "I \otimes S"] \\ \tilde{U}_q & \tilde{U}_q \otimes \tilde{U}_q \arrow[l, "mult"] \end{tikzcd}$$

(ex)

Recall def rels for $\tilde{U}_q$

1. $K_\alpha K_\alpha^{-1} = K_\alpha K_\alpha^{-1} = 1$ $\quad \alpha \in \Pi$

2. $K_\alpha K_\beta = K_\beta K_\alpha$ $\quad \alpha, \beta \in \Pi$

3. $K_\alpha E_\beta K_\alpha^{-1} = q^{(\alpha,\beta)} E_\beta$ $\quad$ --

4. $K_\alpha F_\beta K_\alpha^{-1} = q^{-(\alpha,\beta)} F_\beta$ $\quad$ --

5. $E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha\beta} \dfrac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$ $\quad$ --

Δ is hom

$$\underbrace{\Delta(K_\alpha)}_{K_\alpha \otimes K_\alpha} \underbrace{\Delta(K_\alpha^{-1})}_{K_\alpha^{-1}\otimes K_\alpha^{-1}} \overset{?}{=} 1 \quad \checkmark$$

$$\Delta(K_\alpha)\ \Delta(K_\beta) \overset{?}{=} \Delta(K_\alpha)\ \Delta(K_\beta)$$

$$\begin{aligned} LHS &= (K_\alpha \otimes K_\alpha)(K_\beta \otimes K_\beta) \\ &= K_\alpha K_\beta \otimes K_\alpha K_\beta \\ &= K_\beta K_\alpha \otimes K_\beta K_\alpha \\ &= RHS \end{aligned}$$

$$\Delta(K_\alpha)\ \Delta(E_\beta)\ \Delta(K_\alpha^{-1}) \overset{?}{=} q^{(\alpha,\beta)} \Delta(E_\beta)$$

$$\begin{aligned} LHS &= (K_\alpha \otimes K_\alpha)(E_\beta \otimes 1 + K_\beta \otimes E_\beta)(K_\alpha^{-1} \otimes K_\alpha^{-1}) \\ &= K_\alpha E_\beta K_\alpha^{-1} \otimes 1 + K_\beta \otimes K_\alpha E_\beta K_\alpha^{-1} \\ &= q^{(\alpha,\beta)}(E_\beta \otimes 1 + K_\beta \otimes E_\beta) \\ &= q^{(\alpha,\beta)} \Delta(E_\beta) \\ &= RHS \end{aligned}$$

$$\Delta(K_\alpha)\ \Delta(F_\beta)\ \Delta(K_\alpha^{-1}) \overset{?}{=} q^{-(\alpha,\beta)} \Delta(F_\beta)$$

sim

$$\Delta(E_\alpha)\,\Delta(F_\beta) - \Delta(F_\beta)\,\Delta(E_\alpha) \stackrel{?}{=} \delta_{\alpha\beta}\,\frac{\Delta(K_\alpha) - \Delta(K_\alpha^{-1})}{q_\alpha - q_\alpha^{-1}}$$ 6/22/12

$$LHS = (E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha)(F_\beta \otimes K_\beta^{-1} + 1 \otimes F_\beta)$$

$$- (F_\beta \otimes K_\beta^{-1} + 1 \otimes F_\beta)(E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha)$$

$$= \left\{\begin{array}{l} E_\alpha F_\beta \otimes K_\beta^{-1} \\ + K_\alpha F_\beta \otimes E_\alpha K_\beta^{-1} \\ + E_\alpha \otimes F_\beta \\ + K_\alpha \otimes E_\alpha F_\beta \end{array}\right. - \left\{\begin{array}{l} F_\beta E_\alpha \otimes K_\beta^{-1} \\ + F_\beta K_\alpha \otimes K_\beta^{-1} E_\alpha \\ + E_\alpha \otimes F_\beta \\ + K_\alpha \otimes F_\beta E_\alpha \end{array}\right.$$

Note $K_\alpha F_\beta = q^{-(\alpha,\beta)} F_\beta K_\alpha$

$E_\alpha K_\beta^{-1} = q^{(\alpha,\beta)} K_\beta^{-1} E_\alpha$

so $K_\alpha F_\beta \otimes E_\alpha K_\beta^{-1} = F_\beta K_\alpha \otimes K_\beta^{-1} E_\alpha$

$$= (E_\alpha F_\beta - F_\beta E_\alpha) \otimes K_\beta^{-1} + K_\alpha \otimes (E_\alpha F_\beta - F_\beta E_\alpha)$$

$$= \delta_{\alpha\beta}\,\frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \otimes K_\beta^{-1} + K_\alpha \otimes \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}\,\delta_{\alpha\beta}$$

$$= \delta_{\alpha\beta}\,\frac{K_\alpha \otimes K_\alpha - K_\alpha^{-1} \otimes K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$

$= RHS$

$\underline{\varepsilon \text{ is } \quad \text{alg hom}}$

$$\varepsilon(K_\alpha)\,\varepsilon(K_\alpha^{-1}) \overset{?}{=} 1$$

$$\underset{1}{\overset{\shortparallel}{}} \qquad \underset{1}{\overset{\shortparallel}{}}$$

$$\varepsilon(K_\alpha)\,\varepsilon(K_\beta) \overset{?}{=} \varepsilon(K_\beta)\,\varepsilon(K_\alpha)$$

$$1 = 1$$

$$\varepsilon(K_\alpha)\,\varepsilon(E_\beta)\,\varepsilon(K_\alpha^{-1}) \overset{?}{=} q^{(\alpha,\beta)}\,\varepsilon(E_\beta)$$

$$0 \qquad\qquad 0$$

$$\varepsilon(K_\alpha)\,\varepsilon(F_\beta)\,\varepsilon(K_\alpha^{-1}) \overset{?}{=} q^{-(\alpha,\beta)}\,\varepsilon(F_\beta)$$

$$0 \qquad\qquad 0$$

$$\varepsilon(E_\alpha)\,\varepsilon(F_\beta) - \varepsilon(F_\beta)\,\varepsilon(E_\alpha) \overset{?}{=} \delta_{\alpha\beta}\,\frac{\varepsilon(K_\alpha) - \varepsilon(K_\alpha^{-1})}{q_\alpha - q_\alpha^{-1}}$$

$$0 \qquad\qquad 1-1$$

$\underline{S \text{ is } \quad \text{anti hom}}$

$$S(K_\alpha^{-1})\,S(K_\alpha) \overset{?}{=} 1$$

$$K_\alpha \quad K_\alpha^{-1} \qquad 1$$

$$S(K_\beta)\,S(K_\alpha) \overset{?}{=} S(K_\alpha)\,S(K_\beta)$$

$$K_\beta^{-1} \quad K_\alpha^{-1} \quad \checkmark \quad K_\alpha^{-1} \quad K_\beta^{-1}$$

$$S(K_\alpha^{-1})\,S(E_\beta)\,S(K_\alpha) \overset{?}{=} q^{(\alpha,\beta)}\,S(E_\beta)$$

$$\begin{aligned}
LHS &= K_\alpha\,(-K_\beta^{-1}E_\beta)\,K_\alpha^{-1} \\
&= -K_\beta^{-1}\,K_\alpha E_\beta K_\alpha^{-1} \\
&= -K_\beta^{-1}\,q^{(\alpha,\beta)}\,E_\beta \\
&= q^{(\alpha,\beta)}\,S(E_\beta) \\
&= RHS
\end{aligned}$$

$$S(K_\alpha^{-1})\,S(F_\beta)\,S(K_\alpha) \overset{?}{=} q^{-(\alpha,\beta)}\,S(F_\beta)$$

$$\begin{aligned}
\text{LHS} &= K_\alpha\,(-F_\beta K_\beta)\,K_\alpha^{-1} \\
&= -K_\alpha F_\beta K_\alpha^{-1}\,K_\beta \\
&= -\,q^{-(\alpha,\beta)} F_\beta K_\beta \\
&= q^{-(\alpha,\beta)}\,S(F_\beta) \\
&= \text{RHS}
\end{aligned}$$

$$S(F_\beta)\,S(E_\alpha) - S(E_\alpha)\,S(F_\beta) \overset{?}{=} \delta_{\alpha\beta}\,\frac{S(K_\alpha) - S(K_\alpha^{-1})}{q_\alpha - q_\alpha^{-1}}$$

$$\begin{aligned}
\text{LHS} &= (-F_\beta K_\beta)(-K_\alpha^{-1}E_\alpha) - (-K_\alpha^{-1}E_\alpha)(-F_\beta K_\beta) \\
&= K_\alpha^{-1}K_\beta\left(F_\beta E_\alpha\,q^{(\beta,\beta)-(\alpha,\beta)} - E_\alpha F_\beta\,q^{(\beta,\beta)-(\alpha,\beta)}\right) \\
&= q^{(\beta,\beta)-(\alpha,\beta)}\,K_\alpha^{-1}K_\beta\,\delta_{\alpha\beta}\,\frac{K_\alpha^{-1} - K_\alpha}{q_\alpha - q_\alpha^{-1}} \\
&= \delta_{\alpha\beta}\,\frac{K_\alpha^{-1} - K_\alpha}{q_\alpha - q_\alpha^{-1}} \\
&= \text{RHS}
\end{aligned}$$

Δ is co-assoc

need to commute

$\tilde{U} = \tilde{U}_q$

$$\begin{tikzcd}
 & \tilde{U}\otimes\tilde{U} \arrow[r, "\Delta\otimes 1"] & (\tilde{U}\otimes\tilde{U})\otimes\tilde{U} \arrow[d, "can"] \\
\tilde{U} \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \\
 & \tilde{U}\otimes\tilde{U} \arrow[r, "1\otimes\Delta"] & \tilde{U}\otimes(\tilde{U}\otimes\tilde{U})
\end{tikzcd}$$

check

$$\begin{tikzcd}
 & K_\alpha\otimes K_\alpha \arrow[r] & (K_\alpha\otimes K_\alpha)\otimes K_\alpha \arrow[d, "\checkmark"] \\
K_\alpha \arrow[ur] \arrow[dr] & & \\
 & K_\alpha\otimes K_\alpha \arrow[r] & K_\alpha\otimes(K_\alpha\otimes K_\alpha)
\end{tikzcd}$$

K_α^{-1}, E_α, F_α sim.

$\square$

the diagrams in (v), (vi) readily checked $\checkmark$

<u>Next goal</u> :

describe $S^2 : \widetilde{U_q} \to \widetilde{U_q}$

Recall for $U_q(sl_2)$,

$$S^2(x) = K^{-1} x K \qquad \forall x \in U_q(sl_2)$$

We will find an invertible $T \in \widetilde{U_q}$ st

$$S^2(x) = T^{-1} x T \qquad \forall x \in \widetilde{U_q}$$

(Aside on Lie theory) For our Lie alg $\mathfrak{g}$,

$\forall\ 0 \neq \alpha \in H^*$ def

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \langle \alpha, h \rangle x \quad \forall h \in H \}$$

So $\quad e_i \in \mathfrak{g}_{\alpha_i} \qquad 1 \le i \le n$

$\qquad f_i \in \mathfrak{g}_{-\alpha_i}$

A <u>root</u> of $\mathfrak{g}$ is a nonzero $\alpha \in H^*$ st $\mathfrak{g}_\alpha \neq 0$

Define

$$\Phi = \text{set of roots for } \mathfrak{g}$$

(ex) $\quad \Pi \subseteq \Phi \subseteq \varphi$

(ex) $\quad g = H + \sum_{\alpha \in \Phi} g_\alpha$ $\quad$ (dir sum)

For $\alpha \in \Phi$ write

$$\alpha = \sum_{i=1}^{n} a_i \alpha_i \qquad a_i \in \mathbb{Z}, \quad \alpha_i \in \Pi$$

(ex) Either

$a_i \geq 0 \quad 1 \leq i \leq n$ $\quad$ "α is positive"

or $\quad a_i \leq 0 \quad 1 \leq i \leq n$ $\quad$ "α is negative"

Let

$\Phi^+ =$ set of pos roots for g

$\Phi^- =$ -- neg --

so $\quad \Phi = \Phi^+ \cup \Phi^-$ $\quad$ (disj union)

(ex) $\quad \Phi^- = \{ -\alpha \mid \alpha \in \Phi^+ \}$

Let $\quad n^+ =$ Lie subalg of g gen by $e_1 \dots e_n$

$n^- =$ --- $f_1 \dots f_n$

(ex) $\quad n^+ = \sum_{\alpha \in \Phi^+} g_\alpha \qquad n^- = \sum_{\alpha \in \Phi^-} g_\alpha$

Fix $\alpha \in \Pi$ def lin trans

$$s_\alpha : H^* \longrightarrow H^*$$
$$x \longrightarrow x - \langle \alpha^\vee, x \rangle \alpha$$

(ex) $s_\alpha^2 = I$

so $s_\alpha \in GL(H^*)$
↑ group of all invertible $\mathbb{F}$-linear maps

let $W =$ subgroup of $GL(H^*)$ gen by s_α $\alpha \in \Pi$
"Weyl group of $\mathfrak{g}$"

(ex) $\forall x, y \in H^*$ $\forall w \in W$
$(x, y) = (w(x), w(y))$

(ex) Each of Φ, Δ is W-inv

(ex) $\forall \alpha \in \Pi$
$s_\alpha(\alpha) = -\alpha$
$\Phi^+ \setminus \{\alpha\}$ is s_α-invar
$\Phi^- \setminus \{-\alpha\}$ is s_α-invar