Lecture 21

LEM B We have

(i) $(\alpha_i, \alpha_j) = A_{ij} d_i = A_{ji} d_j \qquad 1 \le i, j \le n$

(ii) $(\alpha_i, \alpha_i) = 2 d_i \qquad 1 \le i \le n$

(iii) $A_{ij} = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \qquad 1 \le i, j \le n$

pf (i) obs

$$(\alpha_i, \alpha_j) = (\nu^{-1}(\alpha_i), \nu^{-1}(\alpha_j))$$

$$= (d_i h_i, d_j h_j)$$

$$= d_i d_j \frac{A_{ij}}{d_j}$$

(ii), (iii) clear $\square$

<u>Notation</u> For $\alpha \in \Pi$ write

$\alpha^\vee = h_i$, where $\alpha = \alpha_i$

From now on assume:

$\mathfrak{g}$ is fixed f.d. s.s. Lie alg / $\mathbb{C}$

$n = \operatorname{rank} \mathfrak{g}$

Cartan matrix A

$\mathbb{F}$ any field char $\mathbb{F} \neq 2$

Fix $0 \neq q \in \mathbb{F}^{\cdot}$ q not root of 1

For $\alpha \in \Pi$ write

$$q_\alpha = q^{d_i} \quad \text{where } \alpha = \alpha_i$$

$\forall\, a \in \mathbb{Z}$ define

$$[a]_\alpha = [a]_{q_\alpha}$$

$$[a]_\alpha^! = [a]_{q_\alpha}^!$$

$$\begin{bmatrix} a \\ r \end{bmatrix}_\alpha = \begin{bmatrix} a \\ r \end{bmatrix}_{q_\alpha}$$

"standard setup"

DEF 14 For $\mathfrak{g}, q$ as above

let $U_q(\mathfrak{g})$ be the $\mathbb{F}$-algebra with gens

$$e_\alpha,\ f_\alpha,\ k_\alpha,\ k_\alpha^{-1} \qquad \alpha \in \Pi$$

and rels:

(i) $k_\alpha k_\alpha^{-1} = k_\alpha^{-1} k_\alpha = 1$ $\qquad \alpha \in \Pi$

(ii) $k_\alpha k_\beta = k_\beta k_\alpha$ $\qquad \alpha, \beta \in \Pi$

(iii) $k_\alpha e_\beta k_\alpha^{-1} = q^{(\alpha,\beta)} e_\beta$ $\qquad$ ''

(iv) $k_\alpha f_\beta k_\alpha^{-1} = q^{-(\alpha,\beta)} f_\beta$ $\qquad$ ''

(v) $e_\alpha f_\beta - f_\beta e_\alpha = \delta_{\alpha\beta} \dfrac{k_\alpha - k_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$ $\qquad$ ''

(vi) $\displaystyle\sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s e_\alpha^{1-A_{\alpha\beta}-s} e_\beta e_\alpha^s = 0$ if $\alpha \neq \beta$ $\qquad$ ''

(vii) $\displaystyle\sum_{s=0}^{1-A_{\alpha\beta}} \begin{bmatrix} 1-A_{\alpha\beta} \\ s \end{bmatrix}_\alpha (-1)^s f_\alpha^{1-A_{\alpha\beta}-s} f_\beta f_\alpha^s = 0$ if $\alpha \neq \beta$ $\qquad$ ''

We often abbr $U_q = U_q(\mathfrak{g})$

DEF 15 Let $\widetilde{U_q(\mathfrak{g})}$ denote the $\mathbb{F}$-algebra

with gens

$$e_\alpha,\ f_\alpha,\ k_\alpha,\ k_\alpha^{-1} \qquad \alpha \in \Pi$$

and rels (i)—(v) in Def 14.

We often abbr

$$\tilde{U}_q = \widetilde{U_q(\mathfrak{g})}$$

Obs $\exists$ surjective algebra hom

$$\tilde{U}_q \to U_q$$

Call (vi), (vii) the <u>q-Serre relations</u>

LEM 16 $\quad \forall \alpha \in \Pi \quad \exists$ algebra hom

$$U_{q_\alpha}(sl_2) \longrightarrow \tilde{U}_q$$

s.t.

$$e \longrightarrow e_\alpha$$
$$f \longrightarrow f_\alpha$$
$$k^{\pm 1} \longrightarrow k_\alpha^{\pm 1}$$

pf $\quad$ Defining rels for $U_{q_\alpha}(sl_2)$:

$$k k^{-1} = k^{-1} k = 1$$
$$k e k^{-1} = q_\alpha^{2} e$$
$$k f k^{-1} = q_\alpha^{-2} f$$
$$e f - f e = \frac{k - k^{-1}}{q_\alpha - q_\alpha^{-1}}$$

In $\tilde{U}_q$

$$k_\alpha k_\alpha^{-1} = k_\alpha^{-1} k_\alpha = 1,$$
$$k_\alpha e_\alpha k_\alpha^{-1} = q^{(\alpha,\alpha)} e_\alpha \qquad q^{(\alpha,\alpha)} = q^{d_i 2}$$
$$= (q_\alpha)^2$$
$$k_\alpha f_\alpha k_\alpha^{-1} = q^{-(\alpha,\alpha)} f_\alpha$$
$$e_\alpha f_\alpha - f_\alpha e_\alpha = \frac{k_\alpha - k_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}}$$

$\square$

We will define Hopf algebra structures on each of $\widetilde{U}_q$, U_q.

This will be done so that the maps

$$\widetilde{U}_q \longrightarrow U_q$$

$$U_{q_\alpha}(sl_2) \longrightarrow \widetilde{U}_q$$

are Hopf alg homomorphisms.

Aside about Hopf algebra homomorphisms

Given two Hopf algebras over $\mathbb{F}$:

$$(A_1, \Delta_1, \varepsilon_1, S_1)$$
$$(A_2, \Delta_2, \varepsilon_2, S_2)$$

Given an alg hom

$$\sigma : A_1 \to A_2$$

want σ to "respect" the above Hopf algebra structures.

To see what is involved consider two A_2 modules M, N

$M \otimes N$ becomes an A_1 module in two ways:

I — Each of M, N is A_1-module via σ

Now $M \otimes N$ is A_1 module via Δ_1

II — $M \otimes N$ is A_2-module via Δ_2

Now $M \otimes N$ is A_1 module via σ

Desire that I, II coincide

Given $x \in A_1$ $\quad m \in M$ $\quad n \in N$

$$\underset{\mathrm{I}}{x \cdot m \otimes n} \overset{?}{=} \underset{\mathrm{II}}{x \cdot m \otimes n}$$

$$\underset{\mathrm{I}}{x \cdot m \otimes n} = \sum_i (x_i \cdot m) \otimes (x_i' \cdot n)$$

$$\Delta_1(x) = \sum_i x_i \otimes x_i'$$

$$= \sum_i (\sigma(x_i) \cdot m) \otimes (\sigma(x_i') \cdot n)$$

$$= (\sigma \otimes \sigma \circ \Delta_1)(x) \quad (m \otimes n)$$

$$\underset{\mathrm{II}}{x \cdot m \otimes n} = \sigma(x) \cdot m \otimes n$$

$$= \Delta_2(\sigma(x)) \cdot m \otimes n$$

$$= (\Delta_2 \circ \sigma)(x) \quad (m \otimes n)$$

Require

$$\sigma \otimes \sigma \circ \Delta_1 = \Delta_2 \circ \sigma$$

In other words this diag commutes:

$$\begin{tikzcd} A_1 \arrow[r, "\sigma"] \arrow[d, "\Delta_1"'] & A_2 \arrow[d, "\Delta_2"] \\ A_1 \otimes A_1 \arrow[r, "\sigma \otimes \sigma"'] & A_2 \otimes A_2 \end{tikzcd}$$

$*$

<u>Aside, cont.</u>

$\mathbb{F}$ becomes A_1 module in two ways:

I $\quad \mathbb{F}$ has "trivial" A_2-module str via ε_2

$\mathbb{F}$ has A_1-module str from this and σ

II $\quad \mathbb{F}$ has "trivial" A_1-module str via ε_1

Desire that I, II coincide

Given $x \in A_1$

$$x \cdot_{\mathrm{I}} 1 \overset{?}{=} x \cdot_{\mathrm{II}} 1$$

$$x \cdot_{\mathrm{I}} 1 = \sigma(x) \cdot 1 = \varepsilon_2(\sigma(x))\, 1$$

$$x \cdot_{\mathrm{II}} 1 = \varepsilon_1(x) 1$$

Require

$$\begin{tikzcd} A_1 \arrow[r, "\sigma"] \arrow[d, "\varepsilon_1"'] & A_2 \arrow[d, "\varepsilon_2"] \\ \mathbb{F} \arrow[r, "\mathrm{id}"'] & \mathbb{F} \end{tikzcd}$$

commutes

~~K k~~

Aside, cont

Given A_2-module M

M^* is A_1-module in two ways:

I $\quad M$ is A_1-module via σ

$\quad M^*$ is A_1-module via S_1

II $\quad M^*$ is A_2-module via S_2

$\quad M^*$ is A_1-module by this and σ

Desire that I, II coincide

Given $x \in A_1$, $\quad m \in M$, $\quad f \in M^*$

$$\underset{\mathrm{I}}{(x.f)\,m} \overset{?}{=} \underset{\mathrm{II}}{(x.f)\,m}$$

$$(x.f)\,m = f(x^{S_1}.m) = f((x^{S_1})^{\sigma}.m)$$

$$(x.f)\,m = (\sigma(x).f)\,m = f((\sigma(x))^{S_2}.m)$$

Require $\quad \sigma \circ S_1 = S_2 \circ \sigma$

In other words this diag commutes:

$$\begin{tikzcd} A_1 \arrow[r, "\sigma"] \arrow[d, "S_1"'] & A_2 \arrow[d, "S_2"] \\ A_1 \arrow[r, "\sigma"'] & A_2 \end{tikzcd}$$

$***$

— o —

DEF 17 With above notation by a hom of Hopf algebras from $(A_1, \Delta_1, \varepsilon_1, S_1)$ to $(A_2, \Delta_2, \varepsilon_2, S_2)$ we mean an algebra hom

$$\sigma : A_1 \to A_2$$

that satisfies $*$, $**$, $***$.

6-21/13

The root Lattice and weight Lattice

Recall $\Pi = \{\alpha_1, \dots, \alpha_n\}$ simple roots

$\alpha_1, \dots, \alpha_n$ basis for H^*

$h_1 \dots h_n$ basis for H

$\langle h_i, \alpha_j \rangle = A_{ij}$ $\quad 1 \le i, j \le n$

DEF 18 Let $\omega_1, \dots, \omega_n$ denote the dual basis for H^* rel $\langle , \rangle : H \times H^* \to \mathbb{C}$

$\langle h_i, \omega_j \rangle = \delta_{ij}$ $\quad 1 \le i, j \le n$

Call ω_i the ith <u>fundamental weight</u>

DEF 19 We set

$$\Lambda = \left\{ \sum_{i=1}^{n} a_i \omega_i \;\middle|\; a_i \in \mathbb{Z} \quad 1 \le i \le n \right\}$$ "weight Lattice"

obs

$$\Lambda = \left\{ \alpha \in H^* \;\middle|\; \langle h_i, \alpha \rangle \in \mathbb{Z} \quad 1 \le i \le n \right\}$$

Set

$$Q = \left\{ \sum_{i=1}^{n} a_i \alpha_i \;\middle|\; a_i \in \mathbb{Z},\ 1 \le i \le n \right\}$$ "root Lattice"

obs

$$Q \subseteq \Lambda$$ since $\langle h_i, \alpha_j \rangle = A_{ij} \in \mathbb{Z} \quad \forall i,j$

LEM 20 We have

(i) $(\alpha_i, \omega_j) = \delta_{ij} d_i$ $(1 \le i, j \le n)$

(ii) $(\alpha, \beta) \in \mathbb{Z}$ $\forall \alpha \in Q$ $\forall \beta \in \Lambda$

pf (i) $(\alpha_i, \omega_j) = \underset{\text{in } H}{\left(\nu^{-1}(\alpha_i), \nu^{-1}(\omega_j)\right)}$

$= \langle \nu^{-1}(\alpha_i), \omega_j \rangle$ def of ν

$= \langle d_i h_i, \omega_j \rangle$

$= \delta_{ij} d_i$

$\square$

(ii) clear from (i)