Lecture 20

CH 3 The Hopf algebra $U_q(\mathfrak{g})$

where $\mathfrak{g}$ = f.d. s.s. Lie algebra

To motivate $U_q(\mathfrak{g})$ for general $\mathfrak{g}$,

first consider $\mathfrak{g} = \mathfrak{sl}_{n+1}$

Recall Lie algebra $\mathfrak{sl}_{n+1}$

For moment assume $\mathbb{F} = \mathbb{C}$

View
$$\mathfrak{sl}_{n+1} = \{ x \in \mathrm{Mat}_{n+1}(\mathbb{F}) \mid \mathrm{tr}(x) = 0 \}$$

Lie bracket
$$[x,y] = xy - yx \qquad x,y \in \mathfrak{sl}_{n+1}$$

Structure of $\mathfrak{sl}_{n+1}$ is described by <u>Cartan matrix</u>

$$A = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix} \quad n \times n$$

as we now explain.

Notation For $1 \le i, j \le n+1$

$E_{ij} \in Mat_{n+1}(\mathbb{F})$

has (i,j)-entry 1 and all other entries 0

Generators for Lie alg $\mathfrak{sl}_{n+1}$:

$1 \le i \le n$

$$e_i = E_{i,i+1}$$

$$f_i = E_{i+1,i}$$

$$h_i = E_{ii} - E_{i+1,i+1}$$

Relations for sl_{n+1}

(i) $[e_i, f_j] = \delta_{ij} h_i \qquad 1 \le i,j \le n$

(ii) $[h_i, h_j] = 0$

(iii) $[h_i, e_j] = A_{ij} e_j$

(iv) $[h_i, f_j] = -A_{ij} f_j$

(v) $(\mathrm{ad}\, e_i)^{1-A_{ij}} (e_j) = 0 \qquad$ if $i \neq j$

(vi) $(\mathrm{ad}\, f_i)^{1-A_{ij}} (f_j) = 0 \qquad$ if $i \neq j$

where $\mathrm{ad}\, x (y) = [x, y]$

By Serre's thm, sl_{n+1} is iso to the Lie algebra over $\mathbb{F}$ generated by symbols e_i, f_i, h_i $(1 \le i \le n)$ subject to relations (i) – (vi) above.

Also the univ enveloping alg $U(sl_{n+1})$ is the assoc algebra with gens e_i, f_i, h_i $(1 \le i \le n)$ and relations (i) – (vi) above, where we interp $[x, y] = xy - yx$

Observations

- For $1 \le i \le n$,

  e_i, f_i, h_i span a Lie subalgebra

  of sl_{n+1} that is iso sl_2

- Ref to (v) above, in $U(sl_{n+1})$ the LHS is

$$\sum_{s=0}^{1-A_{ij}} \binom{1-A_{ij}}{s} (-1)^s e_i^{1-A_{ij}-s} e_j e_i^s$$

DEF 1 Field $\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$ not root of 1

For $n \geq 1$ the algebra $U_q(sl_{n+1})$ has

gens

$$e_i, f_i, k_i, k_i^{-1} \qquad 1 \leq i \leq n$$

and relations

(i) $\quad k_i k_i^{-1} = k_i^{-1} k_i = 1 \qquad 1 \leq i \leq n$

(ii) $\quad k_i k_j = k_j k_i \qquad 1 \leq i, j \leq n$

(iii) $\quad k_i e_j k_i^{-1} = q^{A_{ij}} e_j \qquad$ ″

(iv) $\quad k_i f_j k_i^{-1} = q^{-A_{ij}} f_j \qquad$ ″

(v) $\quad e_i f_j - f_j e_i = \delta_{ij} \dfrac{k_i - k_i^{-1}}{q - q^{-1}} \qquad$ ″

(vi) $\quad \displaystyle\sum_{s=0}^{1-A_{ij}} \begin{bmatrix} 1-A_{ij} \\ s \end{bmatrix}_q (-1)^s e_i^{1-A_{ij}-s} e_j e_i^s = 0 \quad$ if $i \neq j$

(vii) $\quad \displaystyle\sum_{s=0}^{1-A_{ij}} \begin{bmatrix} 1-A_{ij} \\ s \end{bmatrix}_q (-1)^s f_i^{1-A_{ij}-s} f_j f_i^s = 0 \quad$ if $i \neq j$

<u>Next goal</u> Define $U_q(\mathfrak{g})$ for a f.d. ss Lie alg $\mathfrak{g}$

<u>Needed facts on $\mathfrak{g}$</u>

For time being assume $\mathbb{F} = \mathbb{C}$

Recall a f.d. Lie alg $\mathfrak{g}$ over $\mathbb{C}$ is semisimple iff it is a direct sum of simple Lie algebras over $\mathbb{C}$.

The simple Lie algebras over $\mathbb{C}$ are

$$A_n,\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

Let $\mathfrak{g}$ = f.d. ss Lie alg / $\mathbb{C}$.

We associate with $\mathfrak{g}$ a <u>Cartan matrix</u> A.

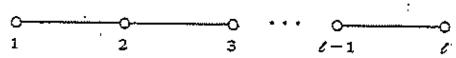
For $\mathfrak{g}$ simple, A is given in handout

For $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r$ ($\mathfrak{g}_i$ simple),

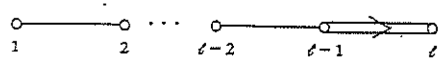
$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_r \end{pmatrix} \qquad A_i = \text{Cartan matrix for } \mathfrak{g}_i$$

By the <u>rank</u> of $\mathfrak{g}$ we mean n where A is $n \times n$

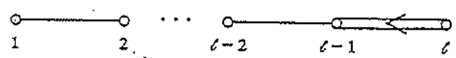
A_ℓ $(\ell \geq 1)$:



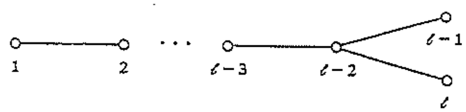
B_ℓ $(\ell \geq 2)$:



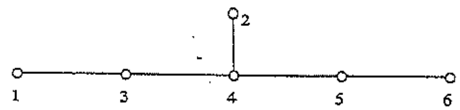
C_ℓ $(\ell \geq 3)$:



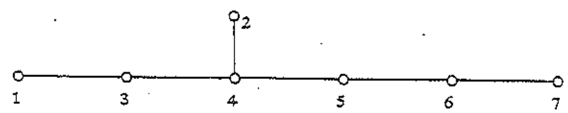
D_ℓ $(\ell \geq 4)$:



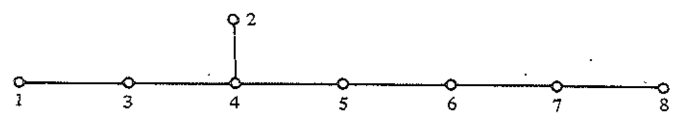
E_6:



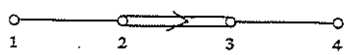
E_7:



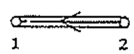
E_8:



F_4:



G_2:



The restrictions on ℓ for types $A_\ell - D_\ell$ are imposed in order to avoid duplication. Relative to the indicated numbering of simple roots, the corresponding Cartan matrices are given in Table 1. Inspection of the diagrams listed above reveals that in all cases except B_ℓ, C_ℓ, the Dynkin diagram can be deduced from the Coxeter graph. However, B_ℓ and C_ℓ both come from a single Coxeter graph, and differ in the relative numbers of short and long simple roots. (These root systems are actually dual to each other, cf. Exercise 5.)

Proof of Theorem. The idea of the proof is to classify first the possible Coxeter graphs (ignoring relative lengths of roots), then see what Dynkin diagrams result. Therefore, we shall merely apply some elementary euclidean geo~~metry~~ to finite sets of vectors whose pairwise angles are those prescribed by [illegible] Coxeter graph. Since we are ignoring lengths, it is easier to work fo[r] the time being with sets of unit vectors. For maximum flexibility, we make

Table 1. *Cartan matrices*

A_ℓ: $\begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & \cdot & \cdot & -1 & 2 \end{pmatrix}$

B_ℓ: $\begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -2 \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix}$

C_ℓ: $\begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & & & \cdot & \cdot & \cdot & 0 & -2 & 2 \end{pmatrix}$

D_ℓ: $\begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & & \cdot & \cdot & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & & & \cdot & \cdot & & -1 & 2 & -1 & -1 \\ 0 & 0 & & & \cdot & \cdot & & 0 & -1 & 2 & 0 \\ 0 & 0 & & & \cdot & \cdot & & 0 & -1 & 0 & 2 \end{pmatrix}$

E_6: $\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$

E_7: $\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$

E_8: $\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$

F_4: $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ G_2: $\begin{pmatrix} 2 & -1 \\ -3 & \end{pmatrix}$

Fact 1 $\det A > 0$. In particular A^{-1} exists.

Fact 2 A is symmetrizable. This means

$\exists$ pos integers $\{d_i\}_{i=1}^{n}$ ($n = \text{rank}\, \mathfrak{g}$)

s/t

$$\begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} A \qquad \text{is symmetric}$$

The d_i are not unique in gen.

We normalize the d_i so that

For $\mathfrak{g}$ simple $\quad 1 = \min\{ d_i \mid 1 \le i \le n \}$

For gen $\mathfrak{g}$ $\quad$ For each simple component of $\mathfrak{g}$ the corresp d_i are normalized as above.

Ex 3

type	$d_1, d_2, \ldots, d_n$
A_n, D_n, E_6, E_7, E_8	$1, 1, \ldots, 1$
B_n	$1, 1, \ldots, 1, 2$
C_n	$2, 2, \ldots 2, 1$
F_4	$1, 1, 2, 2$
G_2	$3, 1$

<u>Fact 4</u> Let $\mathfrak{g}$ = f.d. s.s. Lie alg / $\mathbb{C}$

A = Cartan matrix $n = \text{rank } \mathfrak{g}$

then $\mathfrak{g}$ is iso to the Lie alg / $\mathbb{C}$ with

generators e_i, f_i, h_i $1 \le i \le n$

and rels

$1 \le i, j \le n$

(i) $[e_i, f_j] = \delta_{ij} h_i$

(ii) $[h_i, h_j] = 0$

(iii) $[h_i, e_j] = A_{ij} e_j$

(iv) $[h_i, f_j] = -A_{ij} f_j$

(v) $(\operatorname{ad} e_i)^{1-A_{ij}} (e_j) = 0$ if $i \neq j$

(vi) $(\operatorname{ad} f_i)^{1-A_{ij}} (f_j) = 0$ if $i \neq j$

$\operatorname{ad} x (y) = [x, y]$

<u>Fact 5</u> Given $\mathfrak{g}$ a f.d. s.s. Lie alg / $\mathbb{C}$

$\exists$ nondeg symmetric bilinear form

$$(\, , \,) \quad \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

st
$$([x,y], z) = (x, [y,z]) \qquad \forall x,y,z \in \mathfrak{g}$$

$(\, , \,)$ is not unique in gen. It can be normalized

so that
$$(e_i, f_i) = \frac{1}{d_i} \qquad 1 \le i \le n \qquad n = \text{rank } \mathfrak{g}$$

DEF 6 For g as above define

$$H = \mathrm{Span}(h_1, h_2, \dots, h_n)$$

Obs H is Lie subalg of g

Call H a <u>Cartan subalgebra</u>

Let $H^* =$ dual space of H

We have nondeg bilinear form

$$\langle , \rangle : \quad H \times H^* \to \mathbb{C}$$
$$h \quad f \to f(h)$$

LEM 7 For g, H, H^* as above

$\exists$ unique basis

$$\alpha_1, \alpha_2, \dots, \alpha_n$$

for H^* st

$$\langle h_i, \alpha_j \rangle = A_{ij} \qquad 1 \le i, j \le n$$

$A =$ Cartan matrix

Call $\alpha_1 \dots \alpha_n$ the <u>simple roots</u> of g

Write $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

pf Since A^{-1} exists. $\square$

LEM 8 We have

(i) $$(h_i, h_j) = \frac{A_{ij}}{d_j} = \frac{A_{ji}}{d_i} \qquad 1 \le i, j \le n$$

(ii) $$(h_i, h_i) = \frac{2}{d_i} \qquad 1 \le i \le n$$

(iii) $$A_{ij} = \frac{2(h_i, h_j)}{(h_j, h_j)} \qquad 1 \le i, j \le n.$$

pf (i) $$(h_i, h_j) = ([e_i, f_i], h_j)$$

$$= (e_i, [f_i, h_j])$$

$$= A_{ji}(e_i, f_i)$$

$$= \frac{A_{ji}}{d_i}$$

$$= \frac{A_{ij}}{d_j} \qquad \text{by def of } d_i$$

(ii) Set $j=i$, $A_{ii}=2$ in (i)

(iii) Combine (i), (ii) $\square$

Cor 9 The restriction of $(,)$ to H is nondegenerate.

pf. By LEM 8 (i) and since A^{-1} exists. $\Box$

The following map will be useful.

LEM 10 $\exists$ iso of vector spaces

$$\nu : H \to H^*$$

st $\forall h \in H$

$$\langle h', \nu(h) \rangle = (h', h) \qquad \forall h' \in H$$

pf Each of $\langle , \rangle$, $(,)$ is nondeg. $\Box$

LEM 11 We have

$$\nu(h_i) = \frac{\alpha_i}{d_i} \qquad 1 \le i \le n$$

pf For $1 \le j \le n$ we check:

$$\left\langle h_j, \frac{\alpha_i}{d_i} \right\rangle \stackrel{?}{=} \left\langle h_j, \nu(h_i) \right\rangle$$

$$\left\langle h_j, \frac{\alpha_i}{d_i} \right\rangle = \frac{A_{ji}}{d_i}$$

$$\left\langle h_j, \nu(h_i) \right\rangle = (h_j, h_i) = \frac{A_{ji}}{d_i} \quad \text{LEM 8 (i)}$$

$\square$

Via $\nu : H \to H^*$ we transport the bilinear form $(,)$ on H, to a bilin form $(,)$ on H^*:

Def 12 $\exists$ bilin form

$$(,) : H^* \times H^* \to \mathbb{C}$$

st $(x, y) = (\nu^{-1}(x), \nu^{-1}(y))$ $\forall x, y \in H^*$

in H

We obs $(,)$ is sym, nondeg.