Lecture 2

DEF 7 Let

Λ = subalg of U_q gen by K, K^{-1}

n^+ = subalg of U_q gen by e

n^- = subalg of U_q gen by f

By Thm 5

Λ has basis

$$K^i \qquad i \in \mathbb{Z}$$

n^+ has basis

$$e^i \qquad i \in \mathbb{N}$$

n^- has basis

$$f^i \qquad i \in \mathbb{N}$$

From now on,

- all tensor products are over $\mathbb{F}$

LEM 8 The map

$$U^- \otimes \Lambda \otimes U^+ \to U_q$$ "triangular decomp"

$$x \otimes y \otimes z \mapsto xyz$$

is an isomorphism of vector spaces

pf By thm 5 $\square$

<u>Aside</u> For any algebra A

an <u>automorphism</u> of A is an algebra iso $A \to A$

For example, given an invertible $a \in A$ the map

$$\begin{aligned} A &\to A \\ x &\to a x a^{-1} \end{aligned}$$

is an aut of A, said to be <u>inner</u>

For an alg A, the vector space A with product

$$\begin{aligned} A \times A &\to A \\ a \quad b &\to ba \end{aligned}$$

is an algebra, called the <u>opposite</u> of A and denoted A^{op}

An <u>antiautomorphism</u> of A is an algebra iso $A \to A^{op}$.

Under the composition product,

	aut	antiaut
aut	aut	antiaut
antiaut	antiaut	aut

— o —

Back to U_q.

The following four results are routinely checked.

LEM 9 For $0 \neq \alpha \in \mathbb{F}$ and $n \in \mathbb{Z}$

$\exists$ aut of U_q that sends

$e \to \alpha e k^n$, $f \to \alpha^{-1} k^{-n} f$, $k^{\pm 1} \to k^{\pm 1}$

LEM 10 $\exists$ aut of U_q that sends

$e \to e$, $f \to -f$, $k \to -k$, $k^{-1} \to -k^{-1}$

LEM 11 $\exists$ aut ω of U_q that sends

$e \to f$, $f \to e$, $k \to k^{-1}$, $k^{-1} \to k$

LEM 12 $\exists$ antiaut τ of U_q that sends

$e \to e$, $f \to f$, $k \to k^{-1}$, $k^{-1} \to k$

A $\mathbb{Z}$-grading of U_q

Recall Λ = subalg of U_q gen by k, k^{-1}

By Thm 5

$$U_q = \sum_{r,t \in \mathbb{N}} f^r \Lambda e^t \qquad \text{(direct sum)}$$

For $n \in \mathbb{Z}$ define

$$U_q^{(n)} = \sum_{\substack{r,t \in \mathbb{N} \\ t-r=n}} f^r \Lambda e^t$$

So

$$U_q = \sum_{n \in \mathbb{Z}} U_q^{(n)} \qquad \text{(dir sum)}$$

Observe

$$1 \in U_q^{(0)}$$

$$e \in U_q^{(1)}, \qquad f \in U_q^{(-1)} \qquad k, k^{-1} \in U_q^{(0)}$$

We interpret the $U_q^{(n)}$ as follows

Consider the linear map

$$\begin{aligned} U_q &\longrightarrow U_q \qquad ☆ \\ x &\longmapsto k x k^{-1} \end{aligned}$$

Take $x =$ basis element from Thm 5

Write $x = f^r k^s e^t$

$$\begin{aligned} k x k^{-1} &= k f^r k^s e^t k^{-1} \\ &= \underbrace{(k f^r k^{-1})}_{q^{-2r} f^r} k^s \underbrace{(k e^t k^{-1})}_{q^{2t} e^t} \\ &= q^{2t-2r} x \end{aligned}$$

So x is an eigenvector for ☆ with eigenvalue q^{2t-2r}

This shows

LEM 13 The following holds for $n \in \mathbb{N}$

(i) The map $\maltese$ acts on $U_q^{(n)}$ as $q^{2n} I$

(ii) Assume q is not a root of 1

[means $q^i \neq 1$ for $i = 1, 2, \ldots$]

Then $U_q^{(n)}$ is an eigenspace for $\maltese$ with eigenvalue q^{2n}.

LEM 14 For $n, m \in \mathbb{Z}$,

$$U_q^{(n)} U_q^{(m)} \subseteq U_q^{(n+m)}$$

pf By LEM 2 and the constr,

$$e\, U_q^{(m)} \subseteq U_q^{(m+1)}, \quad f\, U_q^{(m)} \subseteq U_q^{(m-1)}, \quad k\, U_q^{(m)} \subseteq U_q^{(m)}$$

Pick $x \in U_q^{(n)}$

show $x\, U_q^{(m)} \subseteq U_q^{(n+m)}$

WLOG $x = f^r k^s e^t \quad t - r = n$

$$x\, U_q^{(m)} = f^r k^s \underbrace{e^t\, U_q^{(m)}}$$

$$\subseteq U_q^{(m+t)}$$

$$\subseteq U_q^{(m+t)}$$

$$\subseteq U_q^{(m+t-r)}$$

$$= U_q^{(n+m)}$$

$\square$

LEM 15 $U_q^{(0)}$ is a subalgebra of U_q.

Moreover if q is not a root of 1 then

$$U_q^{(0)} = \{ x \in U_q \mid kx = xk \}$$

pf By LEM 13, 14. $\square$

The Casimir element of U_q

Recall: For an algebra A and $x \in A$

x is central whenever $xa = ax$ for all $a \in A$.

The set of central elements in A is a subalgebra of A, called its center and denoted $Z(A)$.

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For U_q observe

$$ef + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = fe + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$$

Let C denote this common value.

"Casimir element"

LEM 16 $\quad C$ is central in U_q

pf check

$$eC \overset{?}{=} Ce$$

$$e\left(fe + \frac{qk + q^{-1}k^{-1}}{(q-q^{-1})^2}\right) = \left(ef + \frac{q^{-1}k + qk^{-1}}{(q-q^{-1})^2}\right)e$$

use $\quad qek = q^{-1}ke$

$\quad\quad\; q^{-1}ek^{-1} = qk^{-1}e$

ok

one similarly checks

$$fC = Cf, \quad kC = Ck, \quad k^{-1}C = Ck^{-1}$$

$\square$

LEM 17 $\quad C$ is fixed by the maps in LEM 9, 11, 12.

C is sent to $-C$ by the map in LEM 10.

pf Routine

$\square$

Recall

$$U_q^{(0)} = \sum_{r \in \mathbb{N}} f^r \Lambda e^r$$

Our next goal is to show that $U_q^{(0)}$ has a basis

$$c^i k^j \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}.$$

LEM 18 For $n \in \mathbb{N}$

$$f^n e^n = \prod_{r=1}^{n} \left(C - \frac{q^{2r-1}K + q^{1-2r}K^{-1}}{(q-q^{-1})^2} \right)$$

pf Ind on n

$n = 0$ ✓

$n \geq 1$:

$$f^n e^n = f^{n-1}(fe)e^{n-1}$$

$$= f^{n-1}\left(C - \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2} \right) e^{n-1}$$

$$= f^{n-1}e^{n-1}\left(C - \frac{q^{2n-1}K + q^{1-2n}K^{-1}}{(q-q^{-1})^2} \right)$$

Result follows.

$\square$

LEM 19 Fix $n \in \mathbb{N}$

$$e^n f^n = \prod_{r=1}^{n} \left(C - \frac{q^{1-2r}K + q^{2r-1}K^{-1}}{(q-q^{-1})^2} \right)$$

pf Apply the aut ω to everything in LEM 18

$\square$

LEM 20 For $n \in \mathbb{N}$ the subspace

$$A + CA + C^2A + \cdots + C^{n-1}A$$

contains

$$f^n e^n - C^n, \qquad e^n f^n - C^n.$$

pf By LEM 18, 19 $\square$

LEM 21 Fa $n \in \mathbb{N}$,

$$\underbrace{\Lambda + C\Lambda + C^2\Lambda + \cdots + C^n\Lambda}_{\mathcal{L}_n} = \underbrace{\Lambda + f\Lambda e + f^2\Lambda e^2 + \cdots + f^n\Lambda e^n}_{\mathcal{R}_n}$$

pf show $\mathcal{L}_n = \mathcal{R}_n$ by ind on n

$n = 0$ ✓

$n \geq 1$: By ind $\mathcal{L}_{n-1} = \mathcal{R}_{n-1}$

$\subseteq$: Fa $i \in \mathbb{Z}$ show

$c^n k^i \in \mathcal{R}_n$

$$c^n k^i = \underbrace{\underbrace{(c^n - f^n e^n)k^i}_{\in\ \mathcal{L}_{n-1} = \mathcal{R}_{n-1}\ \text{(LEM 20)}} + \underbrace{f^n e^n k^i}_{= f^n k^i e^n q^{-2ni} \in f^n\Lambda e^n}}_{\subseteq\ \mathcal{R}_n}$$

2: For $i \in \mathbb{Z}$ show

$$f^n k^i e^n \in \mathcal{L}_n$$

$$f^n k^i e^n = \underbrace{f^n e^n}_{\substack{\text{LEM 18} \\ \mathcal{L}_n}} k^i q^{2ni} \in \mathcal{L}_n$$

$\square$