Lecture 19

L-operators for U_q

L-operators give a variation on the Θ^f construction.

These L-operators are used to obtain solutions to the quantum Yang Baxter equation.

Recall for $x \in U_q$

$$\Delta^{op}(x) = \sum_i x_i' \otimes x_i$$

where $$\Delta(x) = \sum_i x_i \otimes x_i'$$

Recall the U_q-module $\underset{\overset{\shortparallel}{M}}{L(1,1)}$ has a basis u_0, u_1 wrt which

$$e : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f : \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad k : \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

DEF 75 Given a U_q-module N and an F-linear map

$$L : M \otimes N \longrightarrow M \otimes N \qquad M = L(1,1)$$

call L an <u>L-operator</u> whenever

$$\Delta(x) L = L \Delta^{op}(x) \qquad \forall x \in U_q$$

(compare with Lem 57, 67.)

DEF 76 Given a U_q-module N

and an $\mathbb{F}$-linear map

$$L: M \otimes N \to M \otimes N$$

Write

$$M \otimes N = u_0 \otimes N + u_1 \otimes N$$

For $i, j \in \{0, 1\}$ define $L_{ij} \in \mathrm{End}(N)$

s.t. for $v \in N$,

$$L(u_0 \otimes v) = u_0 \otimes L_{00}(v) + u_1 \otimes L_{10}(v)$$
$$L(u_1 \otimes v) = u_0 \otimes L_{01}(v) + u_1 \otimes L_{11}(v)$$

LEM 77 For a U_q-module N, an $\mathbb{F}$-linear map

$$L : M \otimes N \to M \otimes N$$

is an L-operator if and only if

$$k L_{00} = L_{00} k \qquad k L_{01} = q^{-2} L_{01} k$$

$$k L_{10} = q^{2} L_{10} k \qquad k L_{11} = L_{11} k$$

and

$$L_{00} e - q e L_{00} = L_{10} \qquad L_{01} e - q e L_{01} = L_{11} - L_{00} k$$

$$L_{10} e = q^{-1} e L_{10} \qquad q^{-1} e L_{11} - L_{11} e = L_{10} k$$

and

$$f L_{00} - q^{-1} L_{00} f = L_{01} \qquad q L_{01} f = f L_{01}$$

$$f L_{10} - q^{-1} L_{10} f = L_{11} - k^{-1} L_{00} \qquad q L_{11} f - f L_{11} = k^{-1} L_{01}$$

Pf To obtain these equations we chase the vectors

$$u_0 \otimes v, \qquad u_1 \otimes v \qquad v \in N$$

around the diagram implicit in Def 75, using

$$x \in \{e, f, k, k^{-1}\}$$

For instance take $x = k$

$$\begin{array}{ccc}
M \otimes N & \xrightarrow{\Delta^{op}(k) = k \otimes k} & M \otimes N \\
u_0 \otimes v & \longrightarrow & q\, u_0 \otimes kv \\
\downarrow \text{apply } L & & \downarrow \\
 & & q\, u_0 \otimes L_{00} kv + q\, u_1 \otimes L_{10} kv \\
 & & \parallel \\
u_0 \otimes L_{00}(v) + u_1 \otimes L_{10}(v) & \xrightarrow{\Delta(k) = k \otimes k} & q\, u_0 \otimes k L_{00}(v) + q^{-1} u_1 \otimes k L_{10}(v)
\end{array}$$

So

$$L_{00} k = k L_{00}$$

$$q\, L_{10} k = q^{-1} k L_{10}$$

The remaining eqns are similarly obtained

$\square$

Next goal: For the U_q module $N = L(n, \varepsilon)$

we describe the L-operators in more detail.

Fix a basis $\{v_i\}_{i=0}^n$ for N st

$$K v_i = \varepsilon q^{n-2i} v_i \qquad 0 \le i \le n \qquad v_{n+1} = 0$$

$$F v_i = [i+1]_q v_{i+1} \qquad\qquad v_{-1} = 0$$

$$E v_i = \varepsilon [n-i+1]_q v_{i-1}$$

Prop 78 Under A59 an $\mathbb{F}$-linear map

$$L : M \otimes N \to M \otimes N$$

is an L-operator if and only if the maps

$$L_{00}, \quad L_{01}, \quad L_{10}, \quad L_{11} \qquad \ast$$

Look as follows in the basis $\{v_i\}_{i=0}^n$.

Each of $\ast$ is tridiagonal with entries

	$(i, i-1)$	(i, i)	$(i-1, i)$
L_{00}	0	$\dfrac{a q^{n-i} - b q^{i-1}}{q - q^{-1}}$	0
L_{01}	$a q^{n-i}[i]_q$	0	0
L_{10}	0	0	$\varepsilon b q^{i-1}[n-i+1]_q$
L_{11}	0	$\varepsilon \dfrac{a q^{i} - b q^{n-i-1}}{q - q^{-1}}$	0

Here $a, b \in \mathbb{F}$ are arbitrary

Pf

In Lem 77, the first 4 equations imply

$L_{00} v_i \in \mathbb{F} v_i$ $\qquad$ $L_{01} v_i \in \mathbb{F} v_{i+1}$ $\qquad$ $0 \le i \le n$

$L_{10} v_i \in \mathbb{F} v_{i-1}$ $\qquad$ $L_{11} v_i \in \mathbb{F} v_i$

Write

$L_{00} v_i = \alpha_i v_i$ $\qquad$ $L_{01} v_i = \beta_i v_{i+1}$

$L_{10} v_i = \gamma_i v_{i-1}$ $\qquad$ $L_{11} v_i = \delta_i v_i$

Use the last 8 eqns of Lem 77 to solve for the $\alpha_i, \beta_i, \gamma_i, \delta_i$ (ex)

$\square$

Next goal: Given U_q-modules N, V.

use the L-ops for N, V to obtain

L-ops for $N \otimes V$

Thm 7.9 Given U_q-modules N, V

and L-ops

$$L^{(N)} : M \otimes N \to M \otimes N \qquad M = L(1,1)$$

$$L^{(V)} : M \otimes V \to M \otimes V$$

For $i, j \in \{0, 1\}$ define

$$L_{ij} : \sum_{l=0}^{1} L_{il}^{(N)} \otimes L_{lj}^{(V)} \in End(N \otimes V)$$

then the corresp map

$$L : M \otimes (N \otimes V) \to M \otimes (N \otimes V)$$

is an L-op for $N \otimes V$

pf Check L satisfies the eqns in Lem 7.7

For instance, check

$$K L_{10} \overset{?}{=} q^2 L_{10} K \qquad \text{on } N \otimes V$$

$$\text{LHS} = (K \otimes K)\left(L^N_{10} \otimes L^V_{00} + L^N_{11} \otimes L^V_{10} \right)$$

$$= \underbrace{K L^N_{10}}_{= q^2 L^N_{10} K} \otimes \underbrace{K L^V_{00}}_{= L^V_{00} K} + \underbrace{K L^N_{11}}_{= L^N_{11} K} \otimes \underbrace{K L^V_{10}}_{= q^2 L^V_{10} K}$$

$$= q^2 \left(L^N_{10} \otimes L^V_{00} + L^N_{11} \otimes L^V_{10} \right)(K \otimes K)$$

$$= \text{RHS}$$

ok.

$\square$

<u>Next goal</u>

Compare L-operators to Θ^f

Recall

$$\Theta^f = \Theta \circ \tilde{f}$$

Action of Θ^f on $M \otimes N$ $\qquad M = L(1,1)$

is an L-op by Lem 67

Recall

$$\Theta = \sum_{s \in \mathbb{N}} a_s f^s \otimes e^s$$

$$a_s = \frac{(-1)^s (q-q^{-1})^s q^{-\binom{s}{2}}}{[s]_q!} \qquad s \in \mathbb{N}$$

On M,

$$f^2 = 0$$

so on $M \otimes N$

$$\Theta = a_0 I + a_1 f \otimes e$$
$$= I - (q-q^{-1}) f \otimes e$$

Take $N = L(n, \varepsilon)$

basis $\{v_i\}_{i=0}^{n}$ as in Prop 78

wts for M: q, q^{-1}

wts for N: $\varepsilon q^{n}, \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n}$

Normalize $\tilde{f}$ st

$$f(q^{-1}, \varepsilon q^{n}) = 1$$

f:

$\lambda \backslash \mu$	εq^{n}	εq^{n-2}	$\cdots$	εq^{-n}
q	εq^{-n}	εq^{1-n}	$\cdots$	ε
q^{-1}	1	q^{-1}	$\cdots$	q^{-n}

Recall basis u_0, u_1 for M from above Def 75

Write L for the Θ^{f} action on $M \otimes N$

Apply L to $u_0 \otimes v_i$

$$L: \quad u_0 \otimes v_i \xrightarrow[\tilde{f}]{\varepsilon q^{i-n}} u_0 \otimes v_i \xrightarrow[\Theta]{} u_0 \otimes v_i - (q-q^{-1}) \underbrace{f u_0}_{=\,u_1} \otimes \underbrace{e v_i}_{=\,\varepsilon[n-i]_q v_{i+1}}$$

$$L_{00}(v_i) = \varepsilon q^{i-n} v_i$$

$$L_{10}(v_i) = -(q-q^{-1}) q^{i-n} [n-i]_q \; v_{i+1}$$

Apply L to $u_1 \otimes v_i$

$$L: \quad u_1 \otimes v_i \xrightarrow[\tilde{f}]{q^{-i}} u_1 \otimes v_i \xrightarrow[\Theta]{} u_1 \otimes v_i \qquad (f u_1 = 0)$$

$$L_{01} = 0$$

$$L_{11}(v_i) = q^{-i} v_i$$

The above L matches the one in Prop 78 with

$$a = 0$$

$$b = -\varepsilon q^{1-n}(q-q^{-1})$$

$\square$