Lecture 18

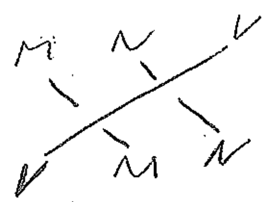
More about Θ^f

Given f.d. U_q-modules M, N, V

Under Assumption 59 we have the U_q-module iso

$$(M \otimes N) \otimes V \longrightarrow V \otimes (M \otimes N)$$
$$u \otimes v \longrightarrow \Theta^f (v \otimes u)$$

Denote this map by

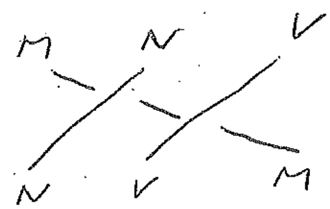


action
↓

We also have the U_q-module iso

$$M \otimes (N \otimes V) \longrightarrow (N \otimes V) \otimes M$$
$$m \otimes u \longrightarrow \Theta^f (u \otimes m)$$

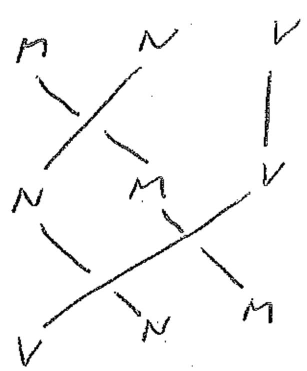
Denote this map by



action
↓

Problems Under A59 show the following for f.d. U_q-modules M, N, V:

↓ action



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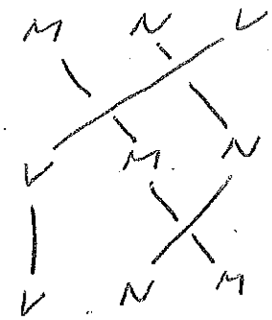
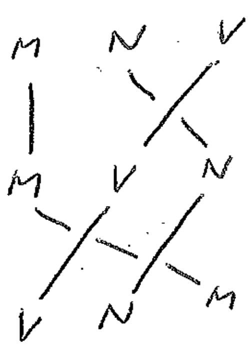


and



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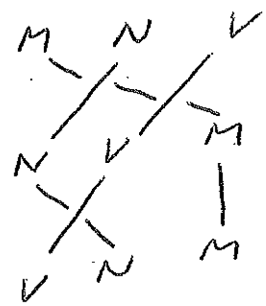
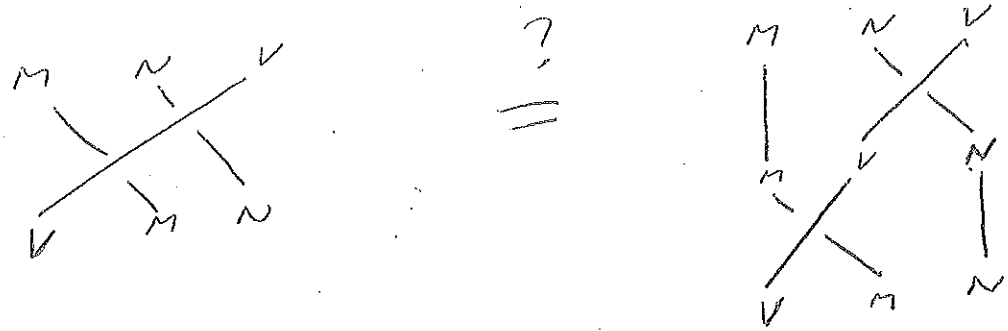


Next goal: Determine if



For $\lambda, \mu, \nu \in \tilde{\Lambda}$

For $m \in M_\lambda$, $n \in N_\mu$, $v \in V_\nu$

apply each side to $m \otimes n \otimes v$

obs $m \otimes n \in (M \otimes N)_{\lambda\mu}$

LHS

$$m \otimes n \otimes v$$

$$\downarrow$$

$$f(\nu, \lambda\mu) \sum_{r\in\mathbb{N}} a_r \, f^r v \otimes e^r(m \otimes n)$$

$$\left[\Delta(e^r) = \sum_{s=0}^{r} q^{s(r-s)} \begin{bmatrix} r \\ s \end{bmatrix}_q e^{r-s} k^s \otimes e^s \right]$$

$$\| $$

$$f(\nu, \lambda\mu) \sum_{r\in\mathbb{N}} \sum_{s=0}^{r} a_r \, q^{s(r-s)} \begin{bmatrix} r \\ s \end{bmatrix}_q f^r v \otimes e^{r-s} k^s m \otimes e^s n$$

[change vars $t = r-s$ so $t \in \mathbb{N}$ and $r = s+t$]

$$\|$$

$$f(\nu, \lambda\mu) \sum_{s\in\mathbb{N}} \sum_{t\in\mathbb{N}} a_{s+t} \, q^{st} \begin{bmatrix} s+t \\ s \end{bmatrix}_q f^{s+t} v \otimes e^t k^s m \otimes e^s n$$

RHS

$$m \otimes n \otimes v$$

$$\Big\downarrow \qquad | \times$$

$$f(\nu,\mu) \sum_{s \in \mathbb{N}} a_s \, m \otimes f^s v \otimes e^s n$$

$$\Big\downarrow \qquad \times |$$

$$f(\nu,u) \sum_{s \in \mathbb{N}} \sum_{t \in \mathbb{N}} a_s \underbrace{f(\nu q^{-2s}, \lambda)}_{\overset{\shortparallel}{f(\nu,\lambda)\lambda^s}} a_t \, f^{s+t} v \otimes e^t m \otimes e^s n$$

$$\shortparallel$$

$$f(\nu,\lambda) \; f(\nu,\mu) \sum_{s \in \mathbb{N}} \sum_{t \in \mathbb{N}} a_s a_t \; f^{s+t} v \otimes e^t k^s m \otimes e^s n$$

LHS RHS always equal provided

$$f(\nu, \lambda\mu) \sum_{s,t \in N} a_{s+t}\, q^{st} \begin{bmatrix} s+t \\ s \end{bmatrix}_q f^{s+t} \otimes e^t k^s \otimes e^s$$

$$=$$

$$f(\nu,\lambda) f(\nu,\mu) \sum_{s,t \in N} a_s a_t\, f^{s+t} \otimes e^t k^s \otimes e^s$$

One checks (ex)

$$a_{s+t}\, q^{st} \begin{bmatrix} s+t \\ s \end{bmatrix}_q = a_s a_t \qquad s, t \in N$$

so the requirement is

$$f(\nu, \lambda\mu) = f(\nu,\lambda) f(\nu,\mu)$$

THM 70 Under A59,

Given f.d. U_q-modules M, N, V,

assume that

$$f(\nu, \lambda\mu) = f(\nu, \lambda) f(\nu, \mu)$$

for all wts

λ of M, μ of N, ν of V.

Then

pf By the discussion above the thm statement. □

Thm 71 Under A59,

Given f.d. U_q-modules M, N, V

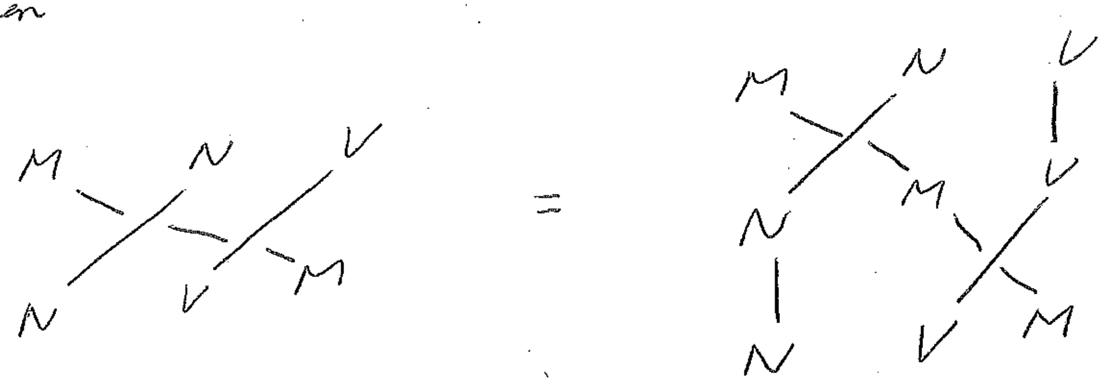
assume that

$$f(\mu\nu, \lambda) = f(\mu, \lambda) f(\nu, \lambda)$$

for all wts

λ of M, μ of N, ν of V.

Then



pf Similar to pf of Thm 70. $\square$

Recall

$$\tilde{\Lambda} = \{ \pm q^i \mid i \in \mathbb{Z} \}$$

Define

$$\tilde{\tilde{\Lambda}} = \{ q^i \mid i \in \mathbb{Z} \}$$

A f.d. U_q-module M has <u>type 1</u> whenever every weight of M is in $\tilde{\tilde{\Lambda}}$

Ex 72.

(i) The trivial U_q-module $\mathbb{F}$ has type 1

(ii) For a f.d. U_q-module M of type 1, then the U_q-module M^* has type 1

(iii) For f.d. U_q modules M, N of type 1, then the U_q-module $M \otimes N$ has type 1.

LEM 73 Under AS9, assume

(i) $f(\nu, \lambda\mu) = f(\nu,\lambda) f(\nu,\mu)$

(ii) $f(\mu\nu, \lambda) = f(\mu, \lambda) f(\nu,\lambda)$

for all $\lambda, \mu, \nu \in \tilde{\tilde{\Lambda}}$

then $f(1,1) = f(q,1) = f(1,q) = 1,$

$$f(q,q)^2 = q^{-1}$$

pf

$$f(1,1) = f(1,1) f(1,1) \quad \text{so}$$
$$f(1,1) = 1$$

$$f(1,q) f(1,q) = f(1,q) \quad \text{so}$$
$$f(1,q) = 1$$

sim $f(q,1) = 1$

$$f(q,q) f(q,q) = f(q^2, q) = f(1,q) q^{-1}$$
$$= q^{-1}$$

$\square$

Comments under A59

assume $\mathbb{F}$ contains $q^{1/2}$

Define

$$f: \begin{array}{ccc} \tilde{\tilde{\Lambda}} \times \tilde{\tilde{\Lambda}} & \longrightarrow & \mathbb{F} \\ q^i \quad q^j & \longrightarrow & q^{-ij/2} \end{array}$$

then for all $\lambda, \mu, \nu \in \tilde{\Lambda}$,

- $f(\lambda,\mu) = f(\lambda q^2, \mu)\mu$
- $f(\lambda,\mu) = f(\lambda, \mu q^2)\lambda$
- $f(\nu, \lambda\mu) = f(\nu,\lambda) f(\nu,\mu)$
- $f(\mu\nu, \lambda) = f(\mu,\lambda) f(\nu,\lambda)$.

Define $\Theta^f = \Theta \circ \tilde{f}$ as before, except now Θ^f acts only on the f.d. U_q-modules of type 1, then Th68, 69, 70, 71 hold for all f.d. U_q-modules M, N, V of type 1.

Problem 74 Under A59,

Given f.d. U_q-modules M, N, M', N'

Given U_q-module homs

$$g: M \longrightarrow M', \qquad h: N \longrightarrow N'$$

Show the following diagram commutes:

$$\begin{tikzcd}
M \otimes N \arrow[r] \arrow[d, "g \otimes h"] & N \otimes M \arrow[d, "h \otimes g"] \\
M' \otimes N' \arrow[r] & N' \otimes M'
\end{tikzcd}$$

$$m \otimes n \longrightarrow \Theta^f(n \otimes m)$$

$$m' \otimes n' \longrightarrow \Theta^f(n' \otimes m')$$