Lecture 17

DEF 64 Under A59, let

$$\underline{\tilde{\Lambda}} = \left\{ \varepsilon q^i \mid i \in \mathbb{Z},\ \varepsilon \in \{1, -1\} \right\}$$

"set of all possible wts for a f.d. U_q-module"

DEF 65 Under A59, take any function

$$f: \tilde{\Lambda} \times \tilde{\Lambda} \to \mathbb{F} \setminus \{0\}$$

s.t. $\forall \lambda, \mu \in \tilde{\Lambda}$

$$f(\lambda,\mu) = f(\lambda q^2, \mu)\mu$$

$$f(\lambda,\mu) = f(\lambda, \mu q^2)\lambda$$

We define an operator $\tilde{f}$ as follows.

$\forall$ f.d. U_q-modules M, N

$$\tilde{f}: M \otimes N \to M \otimes N$$

acts as $f(\lambda,\mu) I$ on $M_\lambda \otimes N_\mu$ $\forall \lambda, \mu \in \tilde{\Lambda}$

Define $\Theta^f = \Theta \circ \tilde{f}$

$\uparrow$

modified R

Ex Ref to Def 65,

for $\lambda, \mu \in \widetilde{\Lambda}$ and $i, j \in \mathbb{Z}$

$$f(\lambda q^{2i}, \mu q^{2j}) = f(\lambda, \mu)\, \lambda^{-j} \mu^{-i} q^{-2ij}$$

So f is uniquely determined by the 16 values

$$f(\lambda, \mu) \qquad \lambda, \mu \in \{1, -1, q, -q\}$$

LEM 66 Under A59,

(i) $(e\otimes 1)\tilde{f} = \tilde{f}(e\otimes k)$

(ii) $(k^{-1}\otimes e)\tilde{f} = \tilde{f}(1\otimes e)$

(iii) $(f\otimes k)\tilde{f} = \tilde{f}(f\otimes 1)$

(iv) $(1\otimes f)\tilde{f} = \tilde{f}(k^{-1}\otimes f)$

(v) $(k\otimes k)\tilde{f} = \tilde{f}(k\otimes k)$

(vi) $(k^{-1}\otimes k^{-1})\tilde{f} = \tilde{f}(k^{-1}\otimes k^{-1})$

pf (i) Given f.d. U_q-modules M, N

Given $m\in M$ $n\in N$

apply each side to $m\otimes n$

Pick $\lambda,\mu\in\tilde{\Lambda}$ WLOG

$m\in M_\lambda$ $n\in N_\mu$

$$e\otimes 1\,\tilde{f}\,m\otimes n = f(\lambda,\mu)\,e\otimes 1\,m\otimes n$$

$$= f(\lambda,\mu)\,em\otimes n$$

Also

$$\tilde{f}(e\otimes k)(m\otimes n) = \tilde{f}(e.m)\otimes(k.n)$$ $k.n=\mu n$

$$= \mu\tilde{f}\,\underbrace{e.m}_{M_{\lambda q^2}}\otimes\underbrace{n}_{N_\mu}$$

$$= \underbrace{\mu f(\lambda q^2,\mu)}_{= f(\lambda,\mu)}\,e.m\otimes n$$ ok.

(ii)–(vi) Sim □

LEM 67 Under A 59,

$$\Delta(x)\,\Theta^f = \Theta^f\,\Delta^{op}(x) \qquad \forall x \in U_q$$

pf WLOG

$$x \in \{e, f, k, k^{-1}\}$$

Case $x=e$

$$\Delta(e)\,\Theta^f = (e\otimes 1 + k\otimes e)\,\Theta\,\tilde{f}$$

$$= \Theta(e\otimes 1 + k^{-1}\otimes e)\,\tilde{f} \qquad \text{Cor 62}$$

$$= \Theta\tilde{f}(e\otimes k + 1\otimes e) \qquad \text{Lem 66}$$

$$= \Theta^f\,\Delta^{op}(e)$$

(extra)

Case $x = f$

$$\Delta(f)\,\Theta^f = (f\otimes k^{-1} + 1\otimes f)\,\Theta\,\tilde{f}$$

$$= \Theta\,(f\otimes k + 1\otimes f)\,\tilde{f}$$

$$= \Theta\,\tilde{f}\,(f\otimes 1 + k^{-1}\otimes f)$$

$$= \Theta^f\,\Delta^{op}(f)$$

Case $x=k$

$$\Delta(k)\,\Theta^f = k\otimes k\,\Theta\,\tilde{f}$$

$$= \Theta\,k\otimes k\,\tilde{f}$$

$$= \Theta\,\tilde{f}\,k\otimes k$$

$$= \Theta^f\,k\otimes k$$

$$= \Theta^f\,\Delta^{op}(k)$$ ✓

Case $x=k^{-1}$ sim

Δ

THM 68 Under A59,

for f.d U_q-modules M, N

the map

$$M \otimes N \longrightarrow N \otimes M$$
$$m \otimes n \longrightarrow \Theta^f (n \otimes m)$$

is an iso of U_q-modules.

pf The map is an iso of VS since it is composition of bijections

$$M \otimes N \longrightarrow N \otimes M \xrightarrow{\tilde{f}} N \otimes M \xrightarrow{\Theta} N \otimes M$$
$$m \otimes n \longrightarrow n \otimes m$$

The map is a hom of U_q-modules; the pf is just like the pf for L57. $\square$

Notation

Recall for f.d. U_q-modules M, N

the map

$$M \otimes N \longrightarrow N \otimes M \qquad \ast$$
$$m \otimes n \longrightarrow \Theta^f (n \otimes m)$$

is U_q-module iso

Denote $\ast$ by

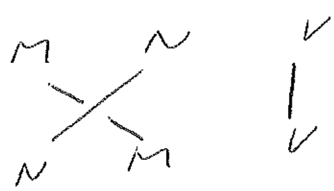
action goes down $\downarrow$

Given U_q-modules M, N, V

the map

$$M \otimes N \otimes V \longrightarrow N \otimes M \otimes V$$
$$\ast \otimes I$$

is denoted

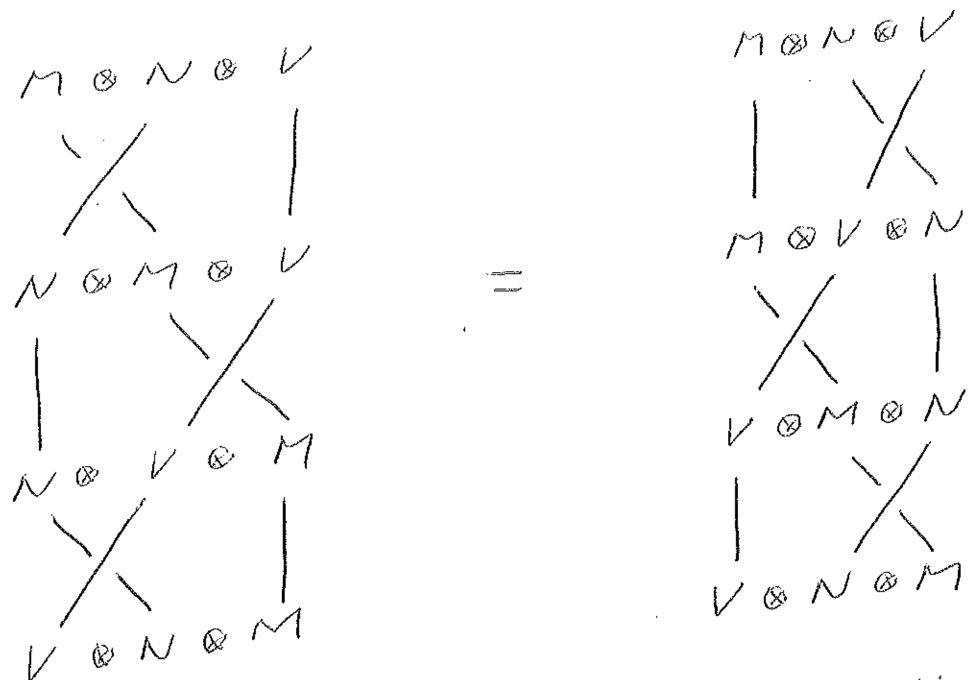


etc.

Theorem 69 Under A59

for f.d. U_q-modules M, N, V,

action $\downarrow$



[For $M = N = V$ this is the quantum Yang-Baxter equation]

pf For $\lambda, \mu, \nu \in \tilde{\Lambda}$.

for $m \in M_\lambda$, $n \in N_\mu$, $v \in V_\nu$

apply each side to $m \otimes n \otimes v$.

$$m \otimes n \otimes v$$

$$\text{X} \, | \qquad \downarrow$$

$$f(\mu,\lambda) \sum_{r \in \mathbb{N}} a_r f^r n \otimes e^r m \otimes v$$

$$| \, \text{X} \qquad \downarrow$$

$$f(\mu,\lambda) \sum_{r,s \in \mathbb{N}} f(\nu, \lambda q^{2r}) a_r a_s f^r n \otimes f^s v \otimes e^{r+s} m$$

$$\text{X} \, | \qquad \downarrow$$

$$f(\mu,\lambda) \sum_{r,s,t \in \mathbb{N}} \underbrace{f(\nu, \lambda q^{2r})}_{\substack{\| \\ f(\nu,\lambda)\nu^{-r}}} \underbrace{f(\nu q^{-2s}, \mu q^{-2r})}_{\substack{\| \\ f(\nu,\mu)\nu^{r}\mu^{s} q^{-2rs}}} a_r a_s a_t f^{s+t} v \otimes e^t f^r n \otimes e^{r+s} m$$

$$\left[\; 1 \otimes k^s \otimes 1 \quad v \otimes n \otimes m = \mu^s \, v \otimes n \otimes m \; \right]$$

$$\|$$

$$f(\mu,\lambda) f(\nu,\lambda) f(\nu,\mu)$$

times

$$\sum_{r,s,t \in \mathbb{N}} a_r a_s a_t q^{-2rs} f^{s+t} \otimes e^t \underbrace{f^r k^s}_{\substack{\| \\ q^{2rs} k^s f^r}} \otimes e^{r+s} \quad (v \otimes n \otimes m)$$

$$m \otimes n \otimes v$$

$$1\,X \qquad \downarrow$$

$$f(\nu,\mu) \sum_{t \in \mathbb{N}} a_t \; m \otimes f^t v \otimes e^t n$$

$$X\,1 \qquad \downarrow$$

$$f(\nu,\mu) \sum_{s,t \in \mathbb{N}} a_s a_t \; f(\nu q^{-2t}, \lambda) \; f^{s+t} v \otimes e^s m \otimes e^t n$$

$$1\,X \qquad \downarrow$$

$$f(\nu,\mu) \sum_{r,s,t \in \mathbb{N}} a_r a_s a_t \; \underbrace{f(\nu q^{-2t}, \lambda)}_{\substack{\shortparallel \\ f(\nu,\lambda)\lambda^t}} \underbrace{f(\mu q^{2t}, \lambda q^{2s})}_{\substack{\shortparallel \\ f(\mu,\lambda)\mu^{-s}\lambda^{-t} q^{-2st}}} \; f^{s+t} v \otimes f^r e^t n \otimes e^{r+s} m$$

$$\shortparallel$$

$$f(\mu,\lambda)\, f(\nu,\lambda)\, f(\nu,\mu)$$

times

$$\sum_{r,s,t \in \mathbb{N}} a_r a_s a_t \, q^{-2st} \; f^{s+t} \otimes f^r \underbrace{e^t k^{-s}}_{\substack{\shortparallel \\ k^{-s} e^t q^{2st}}} \otimes e^{r+s} \cdot (v \otimes n \otimes m)$$

Require

$$\sum_{r,s,t \in \mathbb{N}} a_r a_s a_t \; f^{s+t} \otimes e^t k^s f^r \otimes e^{r+s}$$

$$= \sum_{r,s,t \in \mathbb{N}} a_r a_s a_t \; f^{s+t} \otimes f^r k^{-s} e^t \otimes e^{r+s}$$

Change vars

$$x = r+s$$
$$y = s+t$$

elim r, t

$$r = x - s$$
$$t = y - s$$

$$0 \leq s \leq \min(x,y)$$

Require for $x, y \in \mathbb{N}$,

$$0 = \sum_{s=0}^{\min(x,y)} a_{x-s}\, a_s\, a_{y-s} \left(e^{y-s} k^s f^{x-s} - f^{x-s} k^{-s} e^{y-s} \right)$$

Applying the aut ω and interchange x, y if nec.

WLOG $x \leq y$.

Require for $0 \leq x \leq y$,

$$0 = \sum_{s=0}^{x} a_{x-s}\, a_s\, a_{y-s} \left(e^{y-s} k^s f^{x-s} - f^{x-s} k^{-s} e^{y-s} \right) \qquad \text{☆}$$

All terms are in $U_q^{(n)}$ $\quad n = y - x$

$U_q^{(n)}$ has basis

$$e^{y-x}\, c^i\, k^j \qquad i \in \mathbb{N} \quad j \in \mathbb{Z}$$

To verify ☆, express all terms in above basis

6/7/14

For $0 \le x \le y$, ☆ below

$$0 = \sum_{s=0}^{x} a_{x-s}\, a_s\, a_{y-s} \left(q^{-2s(x-s)} K^{s} \prod_{l=1}^{x-s} \left(C - \frac{q^{1-2l} K + q^{2l-1} K^{-1}}{(q-q^{-1})^2} \right) \right.$$

$$\left. - \; q^{-2s(y-s)} K^{-s} \prod_{l=1}^{x-s} \left(C - \frac{q^{2l-1+2y-2x} K + q^{1-2l-2y+2x} K^{-1}}{(q-q^{-1})^2} \right) \right)$$

☆☆

We next explain how

$\star\star$ is a basic hypergeometric series identity

in disguise.

For $x \in \mathbb{N}$ and indeterminants u, v, w

define

$$ {}_3\varphi_2 \left(\begin{matrix} q^{-x}, & u, & v \\ & 0, & w \end{matrix} \middle| \, q, q \right) = $$

"basic hypergeometric series"

$$ \sum_{n=0}^{x} \frac{(q^{-x};q)_n (u;q)_n (v;q)_n}{(w;q)_n} \frac{q^n}{(q;q)_n} $$

where

$$ (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) $$

One checks

$$ {}_3\varphi_2 \left(\begin{matrix} q^{-x}, & u, & v \\ & 0 & w \end{matrix} \middle| \, q, q \right) = \left(\frac{uv}{w} \right)^x {}_3\varphi_2 \left(\begin{matrix} q^{-x}, & u^{-1}w, & v^{-1}w \\ & 0 & w \end{matrix} \middle| \, q, q \right) $$

Back to ☆☆, view C, K as (commuting) indeterminates and make a change of variables

$$C = \frac{S + S^{-1}}{(q - q^{-1})^2}$$

☆☆ asserts

$${}_3\varphi_2\left(\begin{matrix} q^{-2x}, & u, & v \\ & 0 & w \end{matrix} \middle| q^2, q^2\right) = \left(\frac{uv}{w}\right)^x {}_3\varphi_2\left(\begin{matrix} q^{-2x}, & u^{-1}w, & v^{-1}w \\ & 0 & w \end{matrix} \middle| q^2, q^2\right)$$

where

$$u = qK^{-1}S^{-1}$$

$$v = qK^{-1}S$$

$$w = q^{2y-2x+2}$$

So ☆☆ holds and the result follows. $\square$