Lecture 16

$M \otimes N$ vs $N \otimes M$

Given vector spaces M, N

the map

$$P : \begin{array}{ccc} M \otimes N & \to & N \otimes M \\ m \otimes n & \to & n \otimes m \end{array}$$

is an iso of v.s.

Given Hopf alg $(U_q, \Delta, \varepsilon, S)$

Assume M, N are U_q-modules

P is not a U_q-module iso in general (ex)

Next goal: Determine if the U_q-modules

$M \otimes N$, $N \otimes M$

are iso, and if so find a nice iso between them.

LEM 53 Assume q is not a root of 1 and char $\mathbb{F} \neq 2$

Given f.d. irred U_q-modules

$$L(a,\alpha), \quad L(b,\beta)$$

the following U_q-modules are mutually iso:

(i) $L(a,\alpha) \otimes L(b,\beta)$

(ii) $L(b,\beta) \otimes L(a,\alpha)$

(iii) $\displaystyle\bigoplus_{\substack{i=|a-b| \\ \text{step } 2}}^{a+b} L(i, \alpha\beta)$

pf (i) iso (iii)

$L(a,\alpha) = M$ has a basis $u_0, u_1, \ldots, u_a$ st

$$k . u_i = \alpha q^{a-2i} u_i \qquad 0 \le i \le a$$

$L(b,\beta) = N$ has a basis $v_0, v_1, \ldots, v_b$ st

$$K . v_j = \beta q^{b-2j} v_j \qquad 0 \le j \le b$$

$M \otimes N$ has a basis

$$u_i \otimes v_j \qquad 0 \le i \le a, \quad 0 \le j \le b$$

$$K \cdot (u_i \otimes v_j) = (K \cdot u_i) \otimes (K \cdot v_j) = \alpha \beta q^{a+b-2i-2j} \, u_i \otimes v_j$$

Wts of $M \otimes N$ are

$$\alpha \beta q^{a+b-2l} \qquad 0 \le l \le a+b$$

Corresp sequence of wt space dimensions is

$$1, 2, \dots, r, r, \dots, r, r-1, \dots, 2, 1$$
$$r = \min(a, b)$$

U_q-module (iii) has same wts and wt space dimensions.

So $M \otimes N$ iso (iii) by comments in Lec 7.

(ii) iso (iii) By symmetry

$\square$

COR 54 Assume q not a root of 1 and char $\mathbb{F} \neq 2$.

Given f.d. irred U_q-modules

$L(a,\alpha)$, $L(b,\beta)$, $L(c,\gamma)$

TFAE:

(i) $L(c,\gamma)$ is iso to a submodule of U_q-module $L(a,\alpha) \otimes L(b,\beta)$

(ii) $\alpha\beta\gamma = 1$ and $a+b+c$ even and none of a,b,c is greater than the sum of the other 2 "triangle inequality"

Suppose (i), (ii) hold. Then in (i) the submodule is unique.

pf Reformulation of LEM 53 $\square$

Cor 55 Assume q not root of 1 and char $\mathbb{F} \neq 2$.

For f.d. U_q modules

$$M, N$$

the U_q-modules

$$M \otimes N, \quad N \otimes M$$

are iso.

pf Each f.d. U_q module is s.s

So each of M, N is dir sum of irred U_q modules,

WLOG each of M, N is irred. Result follows

by LEM 53 (i), (ii) $\square$

Next goal: Ref to Cor 55, find a "canonical" iso

of U_q-modules

$$M \otimes N \to N \otimes M$$

DEF 56 Given an algebra A and an algebra hom

$$\Delta : A \to A \otimes A$$

Define

$$\Delta^{op} : A \xrightarrow{\Delta} A \otimes A \xrightarrow{P} A \otimes A$$

$$P(a \otimes b) = b \otimes a$$

So $$\Delta(x) = \sum_i x_i \otimes x_i' \qquad x \in A$$

$$\Delta^{op}(x) = \sum_i x_i' \otimes x_i$$

LEM 57 Given a Hopf alg $(A, \Delta, \varepsilon, S)$

Given $R \in A \otimes A$ st

$$\Delta(x) R = R \Delta^{op}(x) \qquad \forall x \in A$$

then $\forall$ A modules M, N the map

$$M \otimes N \to N \otimes M \qquad *$$

$$m \otimes n \to R.(n \otimes m)$$

is an A module hom.

Moreover if R^{-1} exists then this map is a bijection.

pf Map $*$ is $\mathbb{F}$-linear "

Pick $x \in A$ write

$$\Delta(x) = \sum_i x_i \otimes x_i'$$

So $$\Delta^{op}(x) = \sum_i x_i' \otimes x_i$$

$$\begin{array}{ccc}
 & \ast & \\
M\otimes N & \longrightarrow & N\otimes M \\
m\otimes n & \longrightarrow & R\cdot(n\otimes m)
\end{array}$$

$\downarrow$ apply x

$$x\cdot R\cdot(n\otimes m) = \Delta(x)\,R\,(n\otimes m)$$

$$\begin{array}{ccc}
x\cdot(m\otimes n) = \sum_i (x_i\cdot m)\otimes(x_i'\cdot n) & \longrightarrow & R\cdot\left(\sum_i (x_i'\cdot n)\otimes(x_i\cdot m)\right)
\end{array}$$

$$= R\,\Delta^{op}(x)\,(n\otimes m)$$

Next assume R^{-1} exists.

To show $\ast$ is bijection, display its inverse:

$$\begin{array}{ccc}
N\otimes M & \longrightarrow & M\otimes N \\
n\otimes m & \longrightarrow & P(R^{-1})(m\otimes n)
\end{array}$$

$$P(a\otimes b) = b\otimes a$$

$\square$

<u>Note</u> 58 Given Hopf alg $(A, \Delta, \varepsilon, S)$

and $R \in A \otimes A$

The condition LEM 57 forces R to

be an ∞ sum

$$R = \sum_{i \in \mathbb{N}} r_i \otimes r_i'.$$

whose action on tensor products $M \otimes N$ is not

well defined.

To solve the problem, we interpret R as an

operator that acts on tensor products $M \otimes N$ of

f.d. A-modules M, N.

We focus on $A = U_q$

<u>Assumption 59</u> We assume

(i) q not a root of 1,

(ii) $\operatorname{char} \mathbb{F} \neq 2$

So each f.d. U_q-module is semi simple.

DEF 60 Under Assumption 59 let

$$\Theta_s = a_s f^s \otimes e^s \qquad s \in \mathbb{N}$$

where
$$a_s = \frac{(-1)^s (q-q^{-1})^s q^{-\binom{s}{2}}}{[s]_q^!}$$

Set
$$\Theta_{-1} = 0$$

Def
$$\Theta = \sum_{s \in \mathbb{N}} \Theta_s$$

Note
$$\Theta \notin U_q \otimes U_q$$

We view Θ as an operator that acts on tensor products $M \otimes N$ of f.d. U_q-modules M, N

LEM 61 Under Assumption 59 the following hold for $s \in \mathbb{N}$

(i) $(e\otimes 1)\,\Theta_s + (k\otimes e)\,\Theta_{s-1} = \Theta_s\,(e\otimes 1) + \Theta_{s-1}\,(k^{-1}\otimes e)$

(ii) $(1\otimes f)\,\Theta_s + (f\otimes k^{-1})\,\Theta_{s-1} = \Theta_s\,(1\otimes f) + \Theta_{s-1}\,(f\otimes k)$

(iii) $(k\otimes k)\,\Theta_s = \Theta_s\,(k\otimes k)$

(iv) $(k^{-1}\otimes k^{-1})\,\Theta_s = \Theta_s\,(k^{-1}\otimes k^{-1})$

pf (i) use

$$ef^s - f^s e = [s]_q\, f^{s-1}\,\frac{q^{1-s}k - q^{s-1}k^{-1}}{q-q^{-1}}$$

(ii) Similar

(iii), (iv) Routine

□

COR 62 Under A59,

(i) $(e\otimes 1 + k\otimes e)\,\Theta = \Theta\,(e\otimes 1 + k^{-1}\otimes e)$

(ii) $(1\otimes f + f\otimes k^{-1})\,\Theta = \Theta\,(1\otimes f + f\otimes k)$

(iii) $(k\otimes k)\,\Theta = \Theta\,(k\otimes k)$

(iv) $(k^{-1}\otimes k^{-1})\,\Theta = \Theta\,(k^{-1}\otimes k^{-1})$

$\square$

pf Use Def 60, LEM 61

LEM 63. Under AS9

Given f.d. U_q-modules M, N

(i) $\exists$ basis for $M \otimes N$ wrt which Θ is upper triangular with diagonal entries 1:

$$\Theta : \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ & 0 & & \end{pmatrix}$$

(ii) Θ is invertible on $M \otimes N$

pf (i) M, N are s.s.

WLOG M, N are irred

Write $M = L(a, \alpha)$ $\quad N = L(b, \beta)$

M basis $u_0, u_1, \dots, u_a$ $\quad K \cdot u_i = \alpha q^{a-2i} u_i$

N .. $v_0, v_1, \dots, v_b$ $\quad K \cdot v_j = \beta q^{b-2j} v_j$

$M \otimes N$ basis

$u_i \otimes v_j$ $\quad 0 \le i \le a, \quad 0 \le j \le b$

Order this basis

$u_0 \otimes v_0$, $u_1 \otimes v_0$, $u_0 \otimes v_1$, $u_2 \otimes v_0$, $u_1 \otimes v_1$, $u_0 \otimes v_2$, ..., $u_a \otimes v_b$ $\quad (*)$

Apply Θ to $(*)$.

For $0 \le i \le a$ and $0 \le j \le b$

$$\Theta\, u_i \otimes v_j = \sum_{s \in \mathbb{N}} \Theta_s\, u_i \otimes v_j$$

$$= \sum_{s \in \mathbb{N}} a_s \underbrace{(f^s \cdot u_i)}_{\substack{\uparrow \\ \mathbb{F} u_{i+s} \\ \text{or } 0}} \otimes \underbrace{(e^s \cdot v_j)}_{\substack{\uparrow \\ \mathbb{F} v_{j-s} \\ \text{or } 0}}$$

$$= u_i \otimes v_j + \text{terms to left of } u_i \otimes v_j \text{ in } (*)$$

So wrt $(*)$

$$\Theta : \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

(ii) By (i)

$\square$