Lecture 15

LEM 43 For a Hopf alg $(A, \Delta, \varepsilon, S)$

and $x \in A$

$$\Delta(S(x)) = \sum_i S(x_i') \otimes S(x_i)$$

where
$$\Delta(x) = \sum_i x_i \otimes x_i'$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
 & A \arrow[dl, "S"'] \arrow[r, "\Delta"] & A \otimes A \arrow[d, "S \otimes S"] \\
A \arrow[dr, "\Delta"'] & & A \otimes A \arrow[dl, "P"] \\
 & A \otimes A &
\end{tikzcd}$$

$P(a \otimes b) = b \otimes a$

pf [we will prove this for $A = U_q$ only, as this is what we need, and the pf for general A is difficult]

Suf to check the diagram for $x = e, f, k, k^{-1}$

$\underline{X=e}$ $\qquad \Delta(e) = e\otimes 1 + k\otimes e$

$$\Delta(S(e)) = \Delta(-k^{-1}e)$$
$$= -\Delta(k^{-1})\Delta(e)$$
$$= -(k^{-1}\otimes k^{-1})(e\otimes 1 + k\otimes e)$$
$$= -k^{-1}e\otimes k^{-1} - 1\otimes k^{-1}e$$
$$= S(e)\otimes S(k) + S(1)\otimes S(e)$$

✓

$\underline{x=f}$ $\qquad \Delta(f) = f\otimes k^{-1} + 1\otimes f$

$$\Delta(S(f)) = \Delta(-fk)$$
$$= -\Delta(f)\,\Delta(k)$$
$$= -(f\otimes k^{-1} + 1\otimes f)(k\otimes k)$$
$$= -fk\otimes 1 - k\otimes fk$$
$$= S(f)\otimes S(1) + S(k^{-1})\otimes S(f)$$

✓

$\underline{X=k}$ $\qquad \Delta(k) = k\otimes k$

$$\Delta(S(k)) = \Delta(k^{-1})$$
$$= k^{-1}\otimes k^{-1}$$
$$= S(k)\otimes S(k)$$

✓

$\square$

COR 44 For a Hopf alg $(A, \Delta, \varepsilon, S)$

and A-modules M, N

can: $N^* \otimes M^* \longrightarrow (M \otimes N)^*$

is an A-module hom.

pf By disc above LEM 43 $\square$

One more canonical map

Given vector spaces M, N, V

Compare

$$\mathrm{Hom}(M, \mathrm{Hom}(N, V)) \qquad *$$

$$\mathrm{Hom}(M \otimes N, V) \qquad **$$

next goal: show $*$, $**$ are iso

For $\psi \in *$ $\exists$ $\bar{\psi} \in **$ st

$$\bar{\psi}(m \otimes n) = \psi(m)(n) \qquad \forall m \in M \;\; \forall n \in N$$

LEM 45 With above notation the map

$$\text{can}: \begin{array}{ccc} \mathrm{Hom}(M, \mathrm{Hom}(N, V)) & \to & \mathrm{Hom}(M \otimes N, V) \\ \psi & \to & \bar{\psi} \end{array}$$

is an iso of vector spaces

pf Can is $\mathbb{F}$-linear ✓

Can is inj:

Given $\psi \in X$ st $\bar{\psi} = 0$

$$0 = \bar{\psi}(m \otimes n) \qquad \forall m \in M \ \forall n \in N$$
$$= \psi(m)(n)$$

So $\psi(m) = 0 \qquad \forall \ m \in M$

So $\psi = 0$ ✓

Can is surj:

Given $\varphi \in XX$

find $\psi \in X$ s.t $\bar{\psi} = \varphi$

For $m \in M$ define

$$\psi(m): \begin{array}{ccc} N & \longrightarrow & V \\ n & \longrightarrow & \varphi(m \otimes n) \end{array}$$

then

$$\psi: \begin{array}{ccc} M & \longrightarrow & \mathrm{Hom}(N, V) \\ m & \longrightarrow & \psi(m) \end{array}$$

satisfies $\bar{\psi} = \varphi$

since $\bar{\psi}(m \otimes n) = \psi(m)(n) = \varphi(m \otimes n) \qquad \forall m \in M \ \forall n \in N$

$\square$

THM 46 For a Hopf alg $(A, \Delta, \varepsilon, S)$,

For A-modules M, N, V

the map

$$\text{can}: \quad \mathrm{Hom}(M, \mathrm{Hom}(N, V)) \to \mathrm{Hom}(M \otimes N, V)$$

is an iso of A-modules

pf Given $x \in A$

$$\begin{array}{ccc} \mathrm{Hom}(M, \mathrm{Hom}(N,V)) & \xrightarrow{\text{can}} & \mathrm{Hom}(M \otimes N, V) \\ \varphi & \longrightarrow & \overline{\varphi} \\ & & \downarrow \\ & & x.\overline{\varphi} \\ \downarrow & & \overset{?}{=} \\ x.\varphi & \longrightarrow & \overline{x.\varphi} \end{array}$$

$\downarrow$ apply x

Compare

$x.\overline{\varphi} : M \otimes N \to V$

$\overline{x.\varphi} : M \otimes N \to V$

For $m \in M$, $n \in N$ apply each to $m \otimes n$:

$$(x . \overline{\psi})(m \otimes n) = \sum_i x_i . \left(\overline{\psi}\left(x_i'^{S} . (m \otimes n) \right) \right)$$

$$= \sum_i x_i . \overline{\psi}\left((x_i''^{S} . m) \otimes (x_i'^{S} . n) \right)$$

$$= \sum_i \underbrace{x_i . \underbrace{\left(\underbrace{\left(\psi(\underbrace{x_i''^{S} . m}_{\in M}) \right)}_{\in Hom(N,V)} (\underbrace{x_i'^{S} . n}_{\in N}) \right)}_{\in V}}_{\in V} \qquad *$$

Notation:

$$\begin{tikzcd}
 & x \arrow[d, "\Delta"] & \\
 & \sum_i x_i \otimes x_i' \arrow[dl, "I\otimes\Delta"'] \arrow[dr, "I\otimes S"] & \\
\sum_i x_i \otimes x_i' \otimes x_i'' \arrow[d, "I\otimes S\otimes S"'] & \text{Lem 43} & \sum_i x_i \otimes x_i'^{S} \arrow[ddl, "I\otimes\Delta"] \\
\sum_i x_i \otimes x_i'^{S} \otimes x_i''^{S} \arrow[dr, "I\otimes P"'] & & \\
 & \sum_i x_i \otimes x_i''^{S} \otimes x_i'^{S} &
\end{tikzcd}$$

$$\overline{x, \psi}\,(m \otimes n) = \left(\left(x \cdot \psi\right)(m)\right)(n)$$

$$= \sum_i \left(x_i \cdot \left(\psi\left(x_i'^S \cdot m\right)\right)\right)(n)$$

$$= \sum_i x_i \cdot \left(\psi\left(x_i''^S \cdot m\right)\left(x_i'^S \cdot n\right)\right) \qquad **$$

$*, **$ same ✓

$\square$

DEF 47 Given Hopf alg $(A, \Delta, \varepsilon, S)$

Given A-module M,

define
$$M^A = \{ v \in M \mid x.v = \varepsilon(x)v \quad \forall x \in A \}$$

= subspace of M spanned by the irred A-submodules iso the trivial A-module $\mathbb{F}$

— o —

Next goal: Given Hopf alg $(A, \Delta, \varepsilon, S)$

Given A-modules M, N

Recall A-module
$$\mathrm{Hom}(M, N)$$

Describe
$$\mathrm{Hom}(M, N)^A$$

DEF 48 For any alg A
and A-modules M, N

let
$$\mathrm{Hom}_A(M,N) = \left\{ \varphi \in \mathrm{Hom}(M,N) \;\middle|\; \begin{array}{c} \varphi(x.m) = x.\varphi(m) \\ \forall x \in A \\ \forall m \in M \end{array} \right\}$$

LEM 49 Given Hopf alg $(A, \Delta, \varepsilon, S)$ and A-modules M, N. Then

$$\mathrm{Hom}_A(M,N) \subseteq \mathrm{Hom}(M,N)^A$$

pf Given

$$\varphi \in \mathrm{Hom}_A(M,N)$$

So

$$\varphi(x.m) = x.\varphi(m) \qquad \forall x \in A \ \ \forall m \in M \qquad (*)$$

show

$$x.\varphi = \varepsilon(x)\varphi \qquad \forall x \in A$$

show

$$(x.\varphi)(m) \overset{?}{=} \varepsilon(x)\varphi(m) \qquad \forall x \in A \ \forall m \in M$$

$$\|$$

$$\sum_i x_i \varphi(x_i'^S . m) \qquad \Delta(x) = \sum x_i \otimes x_i'$$

$$\|$$

$$\sum_i x_i (x_i'^S \varphi(m)) \qquad \text{by } *$$

$$\|$$

$$\Big(\underbrace{\underbrace{\sum_i x_i x_i'^S}_{\varepsilon(x) 1_A}\Big) . \varphi(m)}_{\varepsilon(x)\varphi(m)} \qquad N \text{ is } A\text{-module}$$

ok $\square$

THM 50 For the Hopf alg $(U_q, \Delta, \varepsilon, S)$

and U_q modules M, N

$$\mathrm{Hom}_{U_q}(M,N) = \mathrm{Hom}(M,N)^{U_q}$$

pf $\subseteq$ done by LEM 49

$\supseteq$:

Given $\varphi \in \mathrm{Hom}(M,N)^{U_q}$

So $x.\varphi = \varepsilon(x)\varphi$ $\quad x \in U_q$

show $x.\varphi(m) = \varphi(x.m)$ $\quad \forall x \in U_q, \forall m \in M$ $\quad$ ✱

WLOG $x \in \{e, f, k, k^{-1}\}$

$\underline{x = k}$ $\quad \varepsilon(k) = 1$

$$k.\varphi = \varphi$$

$$k.\varphi(m) = \varphi(m) \quad \forall m \in M$$

$$\begin{gathered} k.\varphi(m) \\ \| \\ k\,\varphi(k^S.m) \\ \| \\ k.\varphi(k^{-1}.m) \end{gathered}$$

Gives ✱ for $x = k$

$\underline{x = k^{-1}}$ sim

$\underline{x = e}$

$$\Delta(e) = e\otimes 1 + k\otimes e$$

$$\varepsilon(e) = 0$$

$$S(e) = -k^{-1}e$$

$$e.\varphi = \varepsilon(e)\varphi = 0$$

$\forall m \in M$

$$e.\varphi(m) = 0$$

$$\|$$

$$e.\varphi(1^S.m) + k.\varphi(e^S.m)$$

$$\|$$

$$e.\varphi(m) - k.\varphi(k^{-1}e.m)$$

$$\|$$

$$e.\varphi(m) - \varphi(e.m)$$

(by the case $x = k$)

$\divideontimes$ holds for $x = e$

$x = f$ sim

$\square$

COR 51 For the Hopf alg $(U_q, \Delta, \varepsilon, S)$

Given U_q-modules M, N, V

the iso of U_q-modules

can: $\mathrm{Hom}(M, \mathrm{Hom}(N, V)) \to \mathrm{Hom}(M \otimes N, V)$ (1)

induces an iso of vs

$$\mathrm{Hom}_{U_q}(M, \mathrm{Hom}(N, V)) \to \mathrm{Hom}_{U_q}(M \otimes N, V) \quad (2)$$

pf Since (1) is iso of U_q-modules

it induces an iso of vs

$$\mathrm{Hom}(M, \mathrm{Hom}(N, V))^{U_q} \to \mathrm{Hom}(M \otimes N, V)^{U_q} \quad (3)$$

But LHS of 2 = LHS of 3 by Th 50
RHS of 2 = RHS of 3

Result follows. $\square$

Problems 52 Given Hopf alg $(A, \Delta, \varepsilon, S)$

show the following

- define

$$\Delta' : \quad A \xrightarrow{\Delta} A \otimes A \xrightarrow{P} A \otimes A \qquad P(a \otimes b) = b \otimes a$$

$$\varepsilon' = \varepsilon$$

$$S' = S^{-1}$$

then $(A, \Delta', \varepsilon', S')$ is Hopf alg.

- let σ = aut of A. Define

$$^{\sigma}\Delta : \quad A \xrightarrow{\sigma^{-1}} A \xrightarrow{\Delta} A \otimes A \xrightarrow{\sigma \otimes \sigma} A \otimes A$$

$$^{\sigma}\varepsilon : \quad A \xrightarrow{\sigma^{-1}} A \xrightarrow{\varepsilon} \mathbb{F}$$

$$^{\sigma}S : \quad A \xrightarrow{\sigma^{-1}} A \xrightarrow{S} A \xrightarrow{\sigma} A$$

then $(A, {}^{\sigma}\Delta, {}^{\sigma}\varepsilon, {}^{\sigma}S)$ is Hopf alg.

Let γ = antiaut of A. Define

$$\gamma_\Delta : \quad A \xrightarrow{\gamma^{-1}} A \xrightarrow{\Delta} A \otimes A \xrightarrow{\gamma \otimes \gamma} A \otimes A$$

$$\gamma_\varepsilon : \quad A \xrightarrow{\gamma^{-1}} A \xrightarrow{\varepsilon} \mathbb{F}$$

$$\gamma_S : \quad A \xrightarrow{\gamma^{-1}} A \xrightarrow{S^{-1}} A \xrightarrow{\gamma} A$$

then $(A, \gamma_\Delta, \gamma_\varepsilon, \gamma_S)$ is Hopf alg