

Lecture 14

Lit/11

DEF 36 Under Assumption 31,

and for a f.d. A -module M ,

the quantum trace is the composition

$$\text{tr}_q: \text{End}(M) \xrightarrow{\text{can}^{-1}} M \otimes M^{\vee} \xrightarrow{\text{mof}} \mathbb{F} \rightarrow \text{tr}(k_a^{\vee} m)$$

Note that tr_q is an A -module hom.

Next goal: How is tr_q related to classical trace?

Comments on classical traceGiven $f.d.$ vs M

Consider

$$\text{con: } M \otimes M^* \rightarrow \text{End}(M)$$

$$m \otimes f \rightarrow \varphi_{f,m}$$

Fix basis for M :

$$m_1, m_2, \dots, m_n$$

 M^* has basis

$$m_1^*, m_2^*, \dots, m_n^*$$

"dual basis"

$$\text{sit } m_j^*(m_i) = \delta_{ij}$$

$$1 \leq i, j \leq n$$

 $M \otimes M^*$ has basis

$$m_i \otimes m_j^*$$

$$1 \leq i, j \leq n$$

 $\text{End}(M)$ has basis

$$e_{ij}$$

$$1 \leq i, j \leq n$$

where e_{ij} sends

$$m_l \rightarrow \delta_{l,j} m_i$$

$$1 \leq l \leq n$$

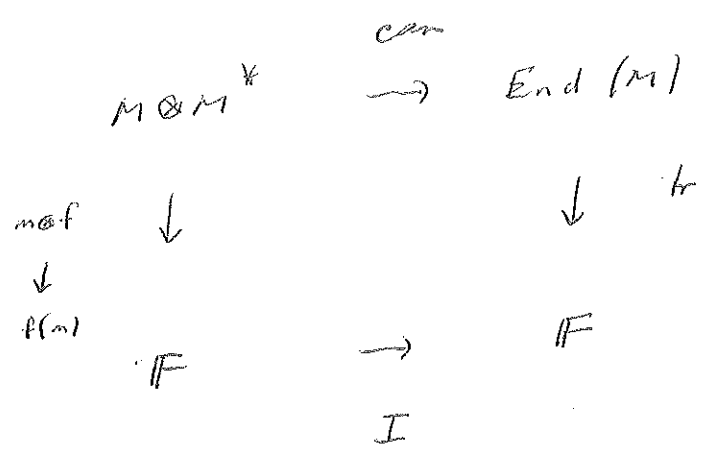
One checks con sends

$$m_i \otimes m_j^* \rightarrow e_{ij}$$

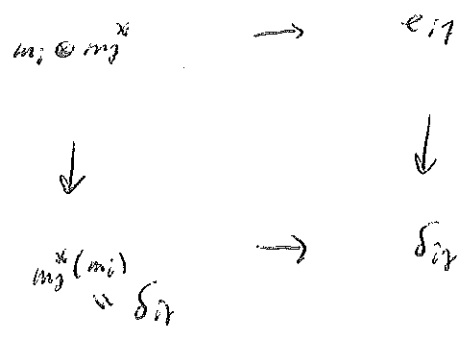
$$1 \leq i, j \leq n$$

LEM 37 With above notation

The following diagram commutes:



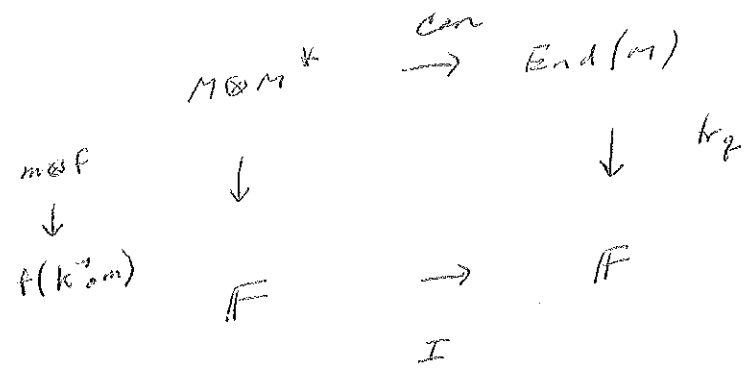
pf F_n is isom



□

Under Assumption 31,

for a f.d. A -module M the following diagram commutes:



Prop 38 Under Assumption 31,
and for a fixed A -module M ,

$$\text{tr}_q(\varphi) = \text{tr}(\varphi \circ k^{-1}) \quad \forall \varphi \in \text{End}(M)$$

↑
composition

pf wlog $\varphi = \varphi_{f,m} \quad m \in M \quad f \in M^*$

So $\text{tr}_q(\varphi) = f(k^{-1} \cdot m)$

claim $k^{-1} \circ \varphi = \varphi_{f, k^{-1} \cdot m}$

pf Apply each side to $n \in M$:

LHS: $n \xrightarrow{\varphi} f(n)m \xrightarrow{k^{-1}} f(n)k^{-1} \cdot m$

RHS: $n \rightarrow f(n)k^{-1} \cdot m$

OK

claim proved ✓

By the claim and Lem 37,

$$\begin{aligned} \text{tr}_q(\varphi) &= f(k^{-1} \cdot m) \\ &= \text{tr}(\varphi_{f, k^{-1} \cdot m}) \\ &= \text{tr}(k^{-1} \circ \varphi) \\ &= \text{tr}(\varphi \circ k^{-1}) \quad (\text{line 27}) \end{aligned}$$

□

Ex 39 Take $A = U_q$ with q not a root of 1.

Take U_q -module $M = L(\lambda, \epsilon)$ $n \in \mathbb{N}$ $\epsilon \in \{1, -1\}$

Recall M has basis

$$m_0, m_1, \dots, m_n$$

st

$$K_0 m_i = \epsilon q^{n-2i} m_i \quad (0 \leq i \leq n)$$

$$f_1 m_i = [i]_q m_{i-1} \quad (0 \leq i \leq n-1)$$

$$f_0 m_n = 0$$

$$e_1 m_i = \epsilon [n-i]_q m_{i+1} \quad (1 \leq i \leq n)$$

$$e_1 m_0 = 0$$

$End(M)$ has corresp basis

$$e_{ij}$$

$$0 \leq i, j \leq n$$

from above LEM 37.

For $0 \leq i, j \leq n$ find

$$tr_q(e_{ii})$$

Sol: Obs

$$k^{-1} = \epsilon \sum_{i=0}^n q^{n-2i} e_{ii}$$

$$\begin{aligned} \text{tr}_g(e_{ij}) &= \text{tr}(e_{ij} \circ k^{-1}) \\ &= \varepsilon \text{tr} \sum_{l=0}^n \underbrace{e_{ij} \circ e_{ll}}_{\delta_{jl} e_{il}} \eta^{2l-n} \end{aligned}$$

$$= \varepsilon \eta^{2j-n} \underbrace{\text{tr} e_{ij}}_{\delta_{ij}}$$

$$= \delta_{ij} \varepsilon \eta^{2j-n}$$



Problem 40

Referring to Ex 39,

- Give the action of e, f, k, k^{-1} on the basis

e_i

$0 \leq i \leq n$

- Show directly that $\mathfrak{h}_{\mathbb{Z}}$ is a $U_{\mathbb{Z}}$ module hom.

More Canonical maps

The map

$$\text{can: } \begin{array}{l} \mathbb{F}^x \rightarrow \mathbb{F} \\ f \rightarrow f(1) \end{array}$$

is an iso of vs.

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Trivial A -module \mathbb{F} satisfies

$$x \cdot \alpha = \varepsilon(x)\alpha$$

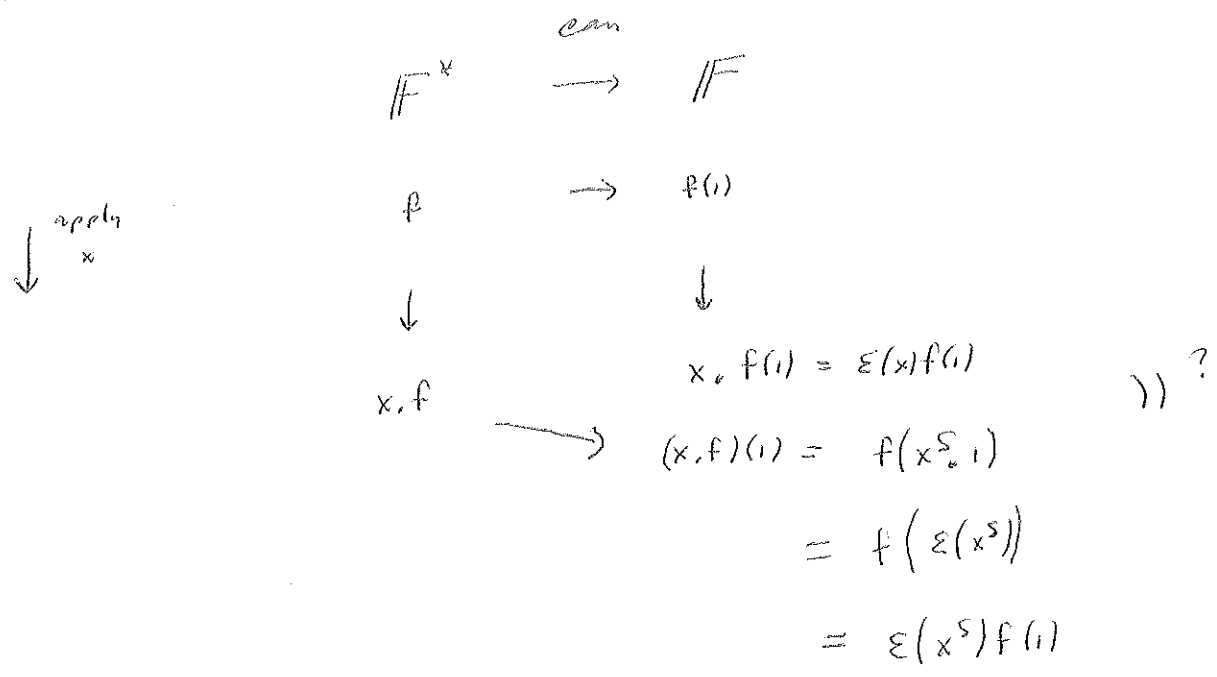
$$x \in A \quad \alpha \in \mathbb{F}$$

\mathbb{F}^x becomes A -module via S

Is $\text{can: } \mathbb{F}^x \rightarrow \mathbb{F}$

an A -module hom?

Pick $x \in A$



Require

$$\mathbb{E}(x^s) = \mathbb{E}(x) \quad \forall x \in A$$

LEM 4) For a Hopf alg $(A, \Delta, \varepsilon, S)$

$$\varepsilon(S(x)) = \varepsilon(x) \quad \forall x \in A$$

pf write

$$\Delta(x) = \sum_i x_i \otimes x_i'$$

Recall $x = \sum_i \varepsilon(x_i) x_i'$

So $S(x) = \sum_i \varepsilon(x_i) S(x_i')$

$$\begin{aligned} \varepsilon(S(x)) &= \sum_i \varepsilon(x_i) \varepsilon(S(x_i')) \\ &= \varepsilon \left(\underbrace{\sum_i x_i S(x_i')}_{\varepsilon(x) 1_A} \right) \end{aligned}$$

$$= \varepsilon(x) \varepsilon(1_A)$$

" 1

$$= \varepsilon(x)$$



COR 92 For a Hopf alg $(A, \Delta, \varepsilon, S)$

the map

$$\text{can}: F^* \rightarrow F$$

is an A -module iso.

□

More canonical maps, cont.

Given vector spaces M, N

Consider the \mathbb{F} -linear map

$$\text{can: } N^* \otimes M^* \rightarrow (M \otimes N)^*$$

$$g \otimes f \rightarrow \overline{g \otimes f}$$

$$\text{where } \overline{g \otimes f}: M \otimes N \rightarrow \mathbb{F}$$

$$m \otimes n \rightarrow f(m)g(n)$$

One checks can is vector space iso if M, N are f.d.

Given Hopf alg (A, Δ, ϵ, S)

Given A -modules M, N

$$\text{Is } \text{can: } N^* \otimes M^* \rightarrow (M \otimes N)^*$$

an A -module hom?

$\forall x \in A$ write $\Delta(x) = \sum_i x_i \otimes x_i'$

$N^* \otimes M^* \xrightarrow{\text{can}} (M \otimes N)^*$

$g \otimes f \rightarrow \overline{g \otimes f}$

↓
x
apply

↓
 $x \circ (g \otimes f)$

↓
 $x \circ \overline{(g \otimes f)}$

$= \sum_i (x_i \circ g) \otimes (x_i' \circ f) \rightarrow \overline{\sum_i (x_i \circ g) \otimes (x_i' \circ f)}$

$x \circ \overline{(g \otimes f)} \stackrel{?}{=} \overline{\sum_i (x_i \circ g) \otimes (x_i' \circ f)}$

For $m \in M$ and $n \in N$, apply each side to $m \otimes n$

RHS:

$$\sum_i \overline{(x_i \circ g) \otimes (x_i' \circ f)} (m \otimes n)$$

$$= \sum_i \left((x_i' \circ f)(m) \right) \left((x_i \circ g)(n) \right)$$

$$= \sum_i f(x_i^S, m) g(x_i^S, n)$$

LHS:

$$\begin{aligned}
 X_0(\overline{g \otimes f})(m \otimes n) &= \overline{g \otimes f}(X^S_{m \otimes n}) \\
 &= \overline{g \otimes f} \left(\sum_j (y_{j,m}) \otimes (y_{j,n}) \right) \\
 &= \sum_j f(y_{j,m}) g(y_{j,n})
 \end{aligned}$$

[write $\Delta(x^S) = \sum y_j \otimes y_j'$]

Require

$$\sum_j y_j \otimes y_j' = \sum_i x_i^S \otimes x_i^S$$

In other words, require

$$\Delta(x^S) = \sum_i x_i^S \otimes x_i^S \quad \forall x \in A$$

LEM 43 For a Hopf alg $(A, \Delta, \varepsilon, S)$,

For $x \in A$,

$$\Delta(S(x)) = \sum_i S(x'_i) \otimes S(x''_i)$$

where

$$\Delta(x) = \sum_i x'_i \otimes x''_i$$

In other words, the following diagram commutes:

$$\begin{array}{ccc}
 & A & \xrightarrow{\Delta} \\
 S \swarrow & & \downarrow \\
 & & A \otimes A \\
 & & \downarrow S \otimes S \\
 A & & A \otimes A \\
 \Delta \searrow & & \downarrow P \\
 & & A \otimes A
 \end{array}
 \quad P(a \otimes b) = b \otimes a$$

pf [We will prove this for $A = U_q$ only, as this is the case we need, and the pf for general A is difficult]

Suf to check the diag for $x = e, f, k, k^{-1}$