Lecture 14

DEF 36 Under Assumption 31, and for a f.d. $A$-module $M$, the quantum trace is the composition

$$\mathrm{tr}_q : \quad \mathrm{End}(M) \xrightarrow[\mathrm{can}^{-1}]{} M \otimes M^* \to \mathbb{F}$$

$$m \otimes f \to f(K_0^{-1} m)$$

Note that $\mathrm{tr}_q$ is an $A$-module hom.

— o —

Next goal: How is $\mathrm{tr}_q$ related to classical trace?

<u>comments on classical trace</u>

Given f.d. vs $M$

Consider

$$\mathrm{can}: \begin{array}{ccc} M \otimes M^* & \to & \mathrm{End}(M) \\ m \otimes f & \to & \varphi_{f,m} \end{array}$$

Fix basis for $M$:

$$m_1, m_2 \ldots m_n$$

$M^*$ has basis

$$m_1^*, m_2^*, \ldots, m_n^* \qquad \text{"dual basis"}$$

s.t. $m_j^*(m_i) = \delta_{ij}$ $\qquad 1 \le i, j \le n$

$M \otimes M^*$ has basis

$$m_i \otimes m_j^* \qquad 1 \le i, j \le n$$

End$(M)$ has basis

$$e_{ij} \qquad 1 \le i, j \le n$$

where $e_{ij}$ sends

$$m_\ell \to \delta_{j,\ell}\, m_i \qquad 1 \le \ell \le n$$

One checks can sends

$$m_i \otimes m_j^* \to e_{ij} \qquad 1 \le i, j \le n$$

LEM 37 With above notation the following diagram commutes:

$$
\begin{tikzcd}
M \otimes M^* \arrow[r, "\mathrm{can}"] \arrow[d, "m\otimes f \,\mapsto\, f(m)"'] & \mathrm{End}(M) \arrow[d, "\mathrm{tr}"] \\
\mathbb{F} \arrow[r, "I"'] & \mathbb{F}
\end{tikzcd}
$$

pf For $1 \le i, j \le n$

$$
\begin{tikzcd}
m_i \otimes m_j^* \arrow[r, mapsto] \arrow[d, mapsto] & e_{ij} \arrow[d, mapsto] \\
m_j^*(m_i) = \delta_{ij} \arrow[r, mapsto] & \delta_{ij}
\end{tikzcd}
$$

$\square$

Under Assumption 31,
for a f.d. $A$-module $M$ the
following diagram commutes:

$$m \otimes f \mapsto f(K_0^{-1} m)$$

$$\begin{tikzcd}
M \otimes M^* \arrow[r, "\mathrm{can}"] \arrow[d] & \mathrm{End}(M) \arrow[d, "\mathrm{tr}_q"] \\
\mathbb{F} \arrow[r, "I"'] & \mathbb{F}
\end{tikzcd}$$

Prop 38 Under Assumption 31, and for a f.d. $A$-module $M$,

$$tr_q(\varphi) = tr(\varphi \circ k^{-1}) \qquad \forall \varphi \in End(M)$$

$\uparrow$ composition

pf WLOG $\varphi = \varphi_{f,m}$ $\quad m \in M \quad f \in M^*$

So $tr_q(\varphi) = f(k^{-1}.m)$

<u>claim</u> $k^{-1} \circ \varphi = \varphi_{f,\, k^{-1}.m}$

<u>pf cl</u> Apply each side to $n \in M$:

LHS: $n \xrightarrow{\varphi} f(n)m \xrightarrow{k^{-1}} f(n)\,k^{-1}.m$

ok

RHS: $n \longrightarrow f(n)\,k^{-1}.m$

claim proved ✓

By the claim and Lem 37,

$$\begin{aligned} tr_q(\varphi) &= f(k^{-1}.m) \\ &= tr(\varphi_{f,\, k^{-1}.m}) \\ &= tr(k^{-1} \circ \varphi) \\ &= tr(\varphi \circ k^{-1}) \quad \text{(Lin alg)} \end{aligned}$$

□

Ex 39 Take $A = U_q$ with $q$ not a root of 1.

Take $U_q$-module $M = L(n, \varepsilon)$ $n \in \mathbb{N}$ $\varepsilon \in \{1, -1\}$

Recall $M$ has basis

$m_0, m_1, \ldots, m_n$

st

$$K \cdot m_i = \varepsilon q^{n-2i} m_i \qquad (0 \le i \le n)$$

$$f \cdot m_i = [i+1]_q m_{i+1} \qquad (0 \le i \le n-1),$$

$$f \cdot m_n = 0$$

$$e \cdot m_i = \varepsilon [n-i+1]_q m_{i-1} \qquad (1 \le i \le n),$$

$$e \cdot m_0 = 0$$

End$(M)$ has corresp basis

$$e_{ij} \qquad 0 \le i, j \le n$$

from above LEM 37.

For $0 \le i, j \le n$ find

$$tr_q(e_{ij})$$

Sol: Obs $$K^{-1} = \varepsilon \sum_{i=0}^{n} q^{n-2i} e_{ii}$$

$$\operatorname{tr}_q(e_{ij}) = \operatorname{tr}(e_{ij} \circ k^{-1})$$

$$= \varepsilon \operatorname{tr} \sum_{\ell=0}^{n} \underbrace{e_{ij} \circ e_{\ell\ell}}_{\delta_{j\ell} e_{i\ell}} q^{2\ell-n}$$

$$= \varepsilon q^{2j-n} \underbrace{\operatorname{tr} e_{ij}}_{\delta_{ij}}$$

$$= \delta_{ij} \varepsilon q^{2j-n}$$

$\square$

L14/8

Problem 40 Referring to Ex 39,

- Give the action of $e, f, k, k^{-1}$ on the basis

  $e_{ij}$ $0 \le i, j \le n$

- Show directly that $tr_q$ is a $U_q$-module hom.

## More Canonical maps

The map

$$\text{can}: \begin{array}{ccc} \mathbb{F}^* & \to & \mathbb{F} \\ f & \to & f(1) \end{array}$$

is an iso of VS.

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Trivial $A$-module $\mathbb{F}$ satisfies

$$x \cdot \alpha = \varepsilon(x)\alpha \qquad x \in A \quad \alpha \in \mathbb{F}$$

$\mathbb{F}^*$ becomes $A$-module via $S$

Is $\quad \text{can}: \mathbb{F}^* \to \mathbb{F}$

an $A$-module hom?

Pick $x \in A$

$$\begin{array}{ccc}
\mathbb{F}^{*} & \xrightarrow{\text{can}} & \mathbb{F} \\
f & \longrightarrow & f(1) \\
\downarrow & & \downarrow \\
 & & x \cdot f(1) = \varepsilon(x) f(1) \quad )) ? \\
x \cdot f & \longrightarrow & (x \cdot f)(1) = f(x^{S} \cdot 1) \\
 & & = f\left(\varepsilon(x^{S})\right) \\
 & & = \varepsilon(x^{S}) f(1)
\end{array}
\qquad \Big\downarrow \text{apply } x$$

Require

$$\varepsilon(x^{S}) = \varepsilon(x) \qquad \forall x \in A$$

LEM 41 For a Hopf alg $(A, \Delta, \varepsilon, S)$

$$\varepsilon(x^S) = \varepsilon(x) \qquad \forall x \in A$$

pf Write

$$\Delta(x) = \sum_i x_i \otimes x_i'$$

Recall
$$x = \sum_i \varepsilon(x_i)\, x_i'$$

So
$$S(x) = \sum_i \varepsilon(x_i)\, S(x_i')$$

$$\varepsilon(S(x)) = \sum_i \varepsilon(x_i)\, \varepsilon(S(x_i'))$$

$$= \varepsilon\Big(\underbrace{\sum_i x_i\, S(x_i')}_{= \varepsilon(x) 1_A}\Big)$$

$$= \varepsilon(x)\, \underbrace{\varepsilon(1_A)}_{=1}$$

$$= \varepsilon(x)$$

$\square$

COR 42 For a Hopf alg $(A, \Delta, \varepsilon, S)$

the map

$$\mathrm{can}: \mathbb{F}^* \to \mathbb{F}$$

is an $A$-module iso. $\square$

# More canonical maps, cont.

Given vector spaces $M, N$

Consider the $\mathbb{F}$-linear map

$$\begin{aligned} N^* \otimes M^* &\longrightarrow (M \otimes N)^* \\ \text{can}: \quad g \otimes f &\longmapsto \overline{g \otimes f} \end{aligned}$$

where

$$\overline{g \otimes f} : \begin{aligned} M \otimes N &\longrightarrow \mathbb{F} \\ m \otimes n &\longmapsto f(m)\, g(n) \end{aligned}$$

One checks can is vector space iso if $M, N$ are f.d.

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Given $A$-modules $M, N$

Is

$$\text{can}: N^* \otimes M^* \to (M \otimes N)^*$$

an $A$-module hom?

$\forall x \in A$ write $\Delta(x) = \sum_i x_i \otimes x_i'$

$$\begin{array}{ccc} N^* \otimes M^* & \xrightarrow{\text{can}} & (M \otimes N)^* \\ g \otimes f & \longrightarrow & \overline{g \otimes f} \\ \Big\downarrow & & \Big\downarrow \\ & & x \circ (\overline{g \otimes f}) \quad \overset{?}{=} \\ x \circ (g \otimes f) & & \\ = \sum_i (x_i \circ g) \otimes (x_i' \circ f) & \longrightarrow & \sum_i \overline{(x_i \circ g) \otimes (x_i' \circ f)} \end{array}$$

$\downarrow$ apply $x$

$$x \circ (\overline{g \otimes f}) \overset{?}{=} \sum_i \overline{(x_i \circ g) \otimes (x_i' \circ f)}$$

For $m \in M$ and $n \in N$, apply each side to

$m \otimes n$

RHS:

$$\sum_i \overline{(x_i \circ g) \otimes (x_i' \circ f)} \; (m \otimes n)$$

$$= \sum_i \big((x_i' \circ f)(m)\big)\big((x_i \circ g)(n)\big)$$

$$= \sum_i f(x_i'^{S} \circ m)\, g(x_i^{S} \circ n)$$

LHS:

$$x \cdot (\overline{g \otimes f})\ (m \otimes n) = \overline{g \otimes f}\,(\,x^S \cdot m \otimes n)$$

$$\left[\text{write } \Delta(x^S) = \sum y_j \otimes y_j' \right]$$

$$= \overline{g \otimes f}\left( \sum_j (y_j \cdot m) \otimes (y_j' \cdot n) \right)$$

$$= \sum_j f(y_j \cdot m)\, g(y_j' \cdot n)$$

Require

$$\sum_j y_j \otimes y_j' = \sum_i x_i'^S \otimes x_i^S$$

In other words, require

$$\Delta(x^S) = \sum_i x_i'^S \otimes x_i^S \qquad \forall x \in A$$

LEM 43 For a Hopf alg $(A, \Delta, \varepsilon, S)$,

Fn $x \in A$,

$$\Delta(S(x)) = \sum_i S(x_i') \otimes S(x_i)$$

where

$$\Delta(x) = \sum_i x_i \otimes x_i'$$

In other words, the following diag commutes:

$$\begin{tikzcd}
 & A \arrow[dl, "S"'] \arrow[dr, "\Delta"] & \\
A \arrow[ddr, "\Delta"'] & & A \otimes A \arrow[d, "S \otimes S"] \\
 & & A \otimes A \arrow[dl, "P"] \\
 & A \otimes A &
\end{tikzcd}$$

$P(a \otimes b) = b \otimes a$

pf [We will prove this for $A = U_q$ only, as this is the case we need, and the pf for general $A$ is difficult]

Suf to check the diag for $x = e, f, k, k^{-1}$