

Lecture 13

LEM 27 For a Hopf alg (A, Δ, ϵ, S)

and A -modules M, N

can: $N \otimes M^* \longrightarrow \text{Hom}(M, N)$

is a hom of $(A \otimes A)$ -modules

Pf By the discussion prior to LEM 26 □

DEF 28 For a Hopf alg (A, Δ, ϵ, S)

and A -modules M, N

We turn $\text{Hom}(M, N)$ from an $(A \otimes A)$ -module into an A -module using Δ .

Concretely, for $x \in A$ write $\Delta(x) = \sum_i x_i \otimes x_i'$

then for $\varphi \in \text{Hom}(M, N)$,

$$\begin{array}{ccc}
 M & \longrightarrow & N \\
 x \cdot \varphi \downarrow & & \\
 m & \longrightarrow & \sum_i x_i \cdot \varphi(x_i' \cdot m)
 \end{array}$$

THM 29 For a Hopf alg $(A, \Delta, \varepsilon, S)$

and A modules M, N the map

$$\text{can} : N \otimes M^* \rightarrow \text{Hom}(M, N)$$

is hom of A -modules.

pf Given $x \in A$ write $\Delta(x) = \sum x_i \otimes x_i'$

$$N \otimes M^* \xrightarrow{\text{can}} \text{Hom}(M, N)$$

$$\begin{array}{ccc} \downarrow \text{apply } x & n \otimes f & \longrightarrow & \varphi_{f, n} \\ & \downarrow & & \downarrow \\ & x \cdot n \otimes f & & x \cdot \varphi_{f, n} = \left(\sum_i x_i \otimes x_i' \right) \cdot \varphi_{f, n} \\ & = \sum_i (x_{i, n}) \otimes (x_i' \cdot f) & \longrightarrow & \sum_i \varphi_{x_i' \cdot f, x_{i, n}} \end{array} \quad \text{)) ?}$$

For $m \in M$ show

$$\left(\sum_i x_i \otimes x_i' \cdot \varphi_{f, n} \right) (m) = \sum_i \varphi_{x_i' \cdot f, x_{i, n}} (m)$$

LHS :

$$\begin{aligned}
 \forall i' \quad (x_i \otimes x_{i'} \circ \varphi_{\text{lin}})(m) &= x_i \varphi_{\text{lin}}(x_{i'}^{iS, m}) \\
 &= x_i \cdot f(x_{i'}^{iS, m}) \cdot n \\
 &= f(x_{i'}^{iS, m}) \cdot x_{i \otimes n}
 \end{aligned}$$

RHS :

$$\begin{aligned}
 \forall i' \quad \varphi_{x_{i \otimes n}, x_{i'}}(m) &= (x_{i \otimes n}^i)(m) \cdot x_{i'} \\
 &= f(x_{i'}^{iS, m}) \cdot x_{i \otimes n}
 \end{aligned}$$

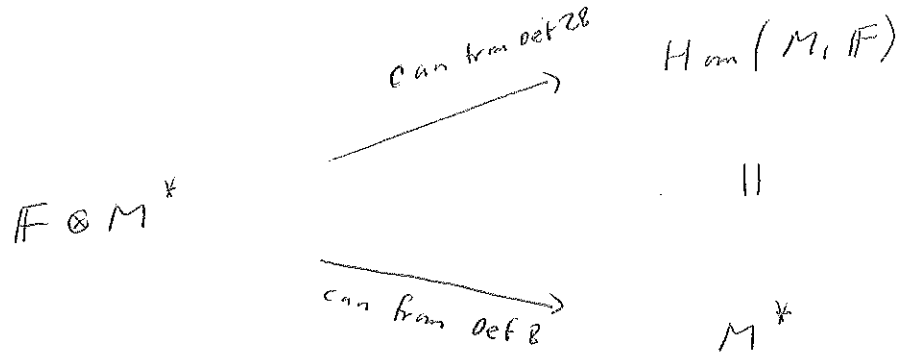
OK

□

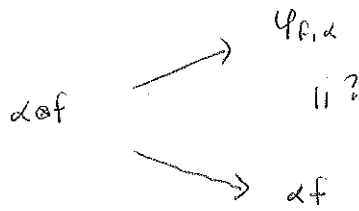
Comments Given Hopf alg (A, Δ, ϵ, S)

Given A module M, N

Take $N = \mathbb{F}$



$$\text{can from def 8} \stackrel{?}{=} \text{can from def 28}$$



$$\begin{array}{ccc}
 \forall m \in M & & \\
 \varphi_{f, \alpha}(m) & \stackrel{?}{=} & (\alpha f)(m) \\
 \parallel & & \parallel \\
 f(m) \alpha & & \alpha f(m) \\
 & & \parallel \\
 & & f(m) \alpha \quad \checkmark
 \end{array}$$

Given Hopf alg $(A, \Delta, \varepsilon, S)$

A -module M . Consider

$$M^* = \text{Hom}(M, \mathbb{F})$$

This has

A -module str from LEM 11

A -module str from DEF 28

(X)

(X')

LEM 30 The A -modules $(*)$, (X') coincide.

pf $\forall x \in A \quad \forall f \in M^* \quad \forall m \in M$

$$\begin{aligned}
 (x, f)(m) & \stackrel{?}{=} (x, f)(m) \\
 * & \stackrel{X'}{=} \sum_i x_i \cdot f(x_i^S, m) \\
 // & \stackrel{X'}{=} \sum_i \varepsilon(x_i) f(x_i^S, m) \\
 f(x^S, m) & \stackrel{X'}{=} f\left(\underbrace{\left(\sum_i \varepsilon(x_i) x_i\right)^S}_x, m\right) \\
 & // \\
 & f(x^S, m)
 \end{aligned}$$

$$\Delta(x) = \sum_i x_i \otimes x_i^S$$

Since $f(x_i^S, m) \in \mathbb{F}$

OK

□

Given V vs M .

For $m \in M$ define a map

$$\bar{m} : \begin{array}{l} M^* \rightarrow \mathbb{F} \\ f \rightarrow f(m) \end{array}$$

Observe that

$$\begin{array}{l} M \rightarrow (M^*)^* \\ m \rightarrow \bar{m} \end{array} \quad \star$$

is \mathbb{F} -linear, and a bijection if $\dim M < \infty$

Call \star the canonical map

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Assume M is A -module

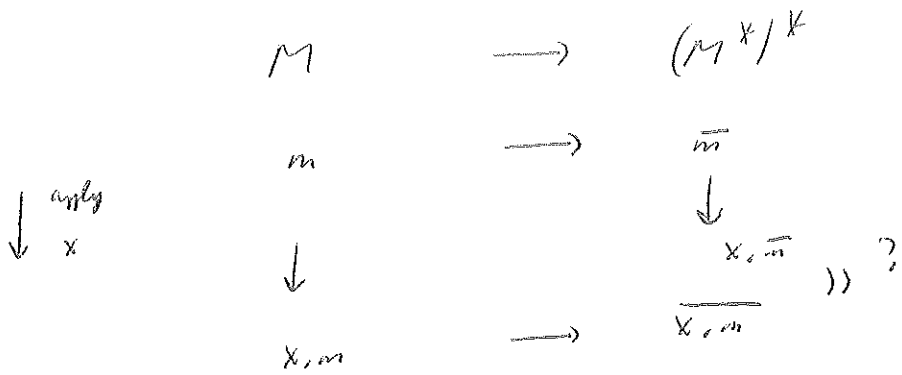
Using S M^* becomes A -module

$$--- (M^*)^* ---$$

next goal: determine if \star is an A -module hom.

Pick $x \in A$

L13/7



For $f \in M^k$

$$(x, \bar{m})(f) \stackrel{?}{=} \overline{x, m}(f)$$

|| \searrow $f(x, m)$

$$\bar{m}(x^S, f)$$

||

$$(x^S, f)(m)$$

||

$$f((x^S)^S, m)$$

Require

$$(x^S)^S = x \quad \forall x \in A$$

ie

$$S^2 = I$$

So \star is not an A -module hom for $A = U_q$

Next goal: show that in some cases we can
 modify \star so it becomes an A -module hom.

Assumption 31 Given Hopf alg $(A, \Delta, \varepsilon, S)$

Assume \exists invertible $k \in A$ st

$$S^2(x) = k^{-1} x k \quad \forall x \in A$$

Note that Assumption holds for $A = U\mathfrak{g}$

LEM 32 Under Assumption 31 the map

$$\begin{aligned} M &\rightarrow (M^*)^* \\ m &\rightarrow \overline{k^{\tau, m}} \end{aligned}$$

$$\bar{m}(+1) = f(m)$$

is an A -module hom

pt For $x \in A$

$$\begin{array}{ccc} M & \rightarrow & (M^*)^* \\ m & \rightarrow & \overline{k^{\tau, m}} \\ \downarrow \text{apply } x & & \downarrow \\ x, m & & x, \overline{k^{\tau, m}} \end{array} \rightsquigarrow \begin{array}{c} \overline{k^{\tau, x, m}} \end{array}$$

For $f \in M^*$

$$\begin{aligned} (x, \overline{k^{\tau, m}})(f) &\stackrel{?}{=} \overline{k^{\tau, x, m}}(f) \\ &\parallel \\ & f(k^{\tau, x, m}) \\ \overline{k^{\tau, m}}(x^S, f) &\parallel \\ & f(k^{\tau} x, m) \\ (x^S, f)(k^{\tau, m}) &\parallel \\ & f((x^S)^S, k^{\tau, m}) \\ &\parallel \\ & f(k^{\tau} x, k^{\tau, m}) \\ &\parallel \\ & f(k^{\tau} x, m) \end{aligned}$$

OK

□

COR 33 For the Hopf alg

$$(U_q, \Delta, \varepsilon, S)$$

and a U_q -module M ,

$$M \longrightarrow (M^*)^*$$

$$m \longrightarrow \overline{k^* \cdot m}$$

is a U_q -module hom.

PF By LEM 32 and since

$$S^2(x) = k^* \cdot x \cdot k$$

$$\forall x \in U_q$$

□

The quantum trace

We will define a quantum trace tr_q for
our Hopf algebras $(U_q, \Delta, \epsilon, S)$

We start with some comments.

Given Hopf algebra (A, Δ, ϵ, S)

Given A -module M

Recall

$$M^* \otimes M \rightarrow \mathbb{F}$$

$$f \otimes m \rightarrow f(m)$$

is an A -module hom

What about

$$M \otimes M^* \rightarrow \mathbb{F}$$

$$m \otimes f \rightarrow f(m)$$

?

Not A -module hom in gen;

$A = U_q$ is
counterexample

LEM 34 Under Assumption 31)

for an A -module M the map

$$M \otimes M^* \rightarrow F$$

$$m \otimes f \rightarrow f(k^{\leftarrow}, m)$$

is an A -module hom.

pf This is the composition

$$M \otimes M^* \rightarrow (M^*)^* \otimes M^* \rightarrow F$$

$$m \otimes f \rightarrow \overline{(k^{\leftarrow}, m)} \otimes f \quad \text{can}$$

↑

↗

A -module hom

□

COR 35 For our Hopf alg

$$(U_q, \Delta, \varepsilon, S)$$

and a U_q -module M the map

$$M \otimes M^* \rightarrow F$$

$$m \otimes f \rightarrow f(k^{-1}m)$$

is a U_q -module hom.

□

Given Hopf alg (A, Δ, ϵ, S)

Given A -modules M, N

Recall

$$N \otimes M^{\vee} \rightarrow \text{Hom}(M, N)$$

$$\text{can: } n \otimes f \rightarrow \varphi_{f,n} \quad \varphi_{f,n}(m) = f(m)n$$

is A -module hom.

Special case $N = M$

$$\text{can: } M \otimes M^{\vee} \rightarrow \text{Hom}(M, M) = \text{End}(M)$$

is A -module hom.

More special case

$$\dim M < \infty$$

$$\text{can: } M \otimes M^{\vee} \rightarrow \text{End}(M)$$

is bijection and hence iso of A -modules