Lecture 13

LEM 27 For a Hopf alg $(A, \Delta, \varepsilon, S)$

and A-modules M, N

can: $N \otimes M^* \longrightarrow Hom(M, N)$

is a hom of $(A \otimes A)$-modules

pf By the discussion prior to LEM 26 $\square$

DEF 28 For a Hopf alg $(A, \Delta, \varepsilon, S)$

and A-modules M, N

we turn $Hom(M,N)$ from an $(A \otimes A)$-module

into an A-module using Δ.

Concretely, for $x \in A$ write $\Delta(x) = \sum_i x_i \otimes x_i'$

then for $\varphi \in Hom(M,N)$,

$$x \cdot \varphi : \begin{array}{ccc} M & \longrightarrow & N \\ m & \longmapsto & \sum_i x_i \cdot \varphi(x_i'^S \cdot m) \end{array}$$

THM 29 For a Hopf alg $(A, \Delta, \varepsilon, S)$ and A-modules M, N the map

$$\mathrm{Can}: \quad N \otimes M^* \longrightarrow \mathrm{Hom}(M, N)$$

is hom of A-modules.

pf Given $x \in A$ write $\Delta(x) = \sum x_i \otimes x_i'$

$$\begin{array}{ccc} N \otimes M^* & \xrightarrow{\mathrm{Can}} & \mathrm{Hom}(M,N) \\ n \otimes f & \longrightarrow & \varphi_{f,n} \\ \Big\downarrow & & \Big\downarrow \\ & & x \cdot \varphi_{f,n} = \left(\sum_i x_i \otimes x_i'\right) \cdot \varphi_{f,n} \\ x \cdot n \otimes f & & \overset{?}{=} \\ = \sum_i (x_i \cdot n) \otimes (x_i' \cdot f) & \longrightarrow & \sum_i \varphi_{x_i' \cdot f,\, x_i \cdot n} \end{array}$$

$\big\downarrow$ apply x

For $m \in M$ show

$$\left(\sum_i x_i \otimes x_i' \cdot \varphi_{f,n}\right)(m) = \sum_i \varphi_{x_i' \cdot f,\, x_i \cdot n}(m)$$

LHS :

$$\forall i \quad \left(x_i \otimes x_i' \circ \varphi_{f,n}\right)(m) = x_i \, \varphi_{f,n}\left(x_i'^{S} \circ m\right)$$

$$= x_i \circ f\left(x_i'^{S} \circ m\right) n$$

$$= f\left(x_i'^{S} \circ m\right) x_i \circ n$$

RHS :

$$\forall i \quad \varphi_{x_i' \circ f,\, x_i \circ n}(m) = \left(x_i' \circ f\right)(m) \cdot x_i \circ n$$

$$= f\left(x_i'^{S} \circ m\right) x_i \circ n$$

ok $\square$

Comments Given Hopf alg $(A, \Delta, \varepsilon, S)$

Given A-modules M, N

Take $N = \mathbb{F}$

$$\begin{tikzcd} & \mathrm{Hom}(M, \mathbb{F}) \arrow[d, equal] \\ \mathbb{F} \otimes M^* \arrow[ur, "\text{can from Def 28}"] \arrow[r, "\text{can from Def 8}"'] & M^* \end{tikzcd}$$

$$\text{can from Def 8} \overset{?}{=} \text{can from Def 28}$$

$$\begin{tikzcd} & \varphi_{f,\alpha} \arrow[d, equal, "?"] \\ \alpha \otimes f \arrow[ur] \arrow[r] & \alpha f \end{tikzcd}$$

$\forall m \in M$

$$\begin{array}{ccc} \varphi_{f,\alpha}(m) & \overset{?}{=} & (\alpha f)(m) \\ \| & & \| \\ f(m)\alpha & & \alpha f(m) \\ & & \| \\ & & f(m)\alpha \end{array}$$

✓

Given Hopf alg $(A, \Delta, \varepsilon, S)$

A-module M. Consider

$$M^* = \mathrm{Hom}(M, \mathbb{F})$$

This has

A-module str from LEM 11 (*)

A-module str from DEF 28 (**)

LEM 30 The A-modules (*), (**) coincide.

pf $\forall x \in A \quad \forall f \in M^* \quad \forall m \in M$

$$\underset{*}{(x.f)(m)} \overset{?}{=} \underset{**}{(x.f)(m)}$$

$$(x.f)(m) \overset{*}{=} f(x^S.m)$$

$$(x.f)(m) \overset{**}{=} \sum_i x_i . f(x_i'^S . m) \qquad \Delta(x) = \sum_i x_i \otimes x_i'$$

$$= \sum_i \varepsilon(x_i) f(x_i'^S . m) \qquad \text{Since } f(x_i'^S m) \in \mathbb{F}$$

$$= f\Big(\big(\underbrace{\textstyle\sum_i \varepsilon(x_i) x_i'}_{x}\big)^S . m\Big)$$

$$= f(x^S . m)$$

ok

$\square$

Given vs M.

For $m \in M$ define a map

$$\bar{m} : \begin{array}{ccc} M^* & \longrightarrow & \mathbb{F} \\ f & \longrightarrow & f(m) \end{array}$$

Obs that

$$\begin{array}{ccc} M & \longrightarrow & (M^*)^* \\ m & \longrightarrow & \bar{m} \end{array} \qquad ☆$$

is $\mathbb{F}$-linear, and a bijection if $\dim M < \infty$

Call ☆ the <u>canonical map</u>

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Assume M is A-module

Using S M^* becomes A-module

- - - $(M^*)^*$ - - -

Next goal: determine if ☆ is an A-module hom.

Pick $x \in A$

$$\begin{array}{ccc} M & \longrightarrow & (M^*)^* \\ m & \longrightarrow & \overline{m} \\ \downarrow & & \downarrow \\ & & x.\overline{m} \quad \overset{?}{\text{''}} \\ x.m & \longrightarrow & \overline{x.m} \end{array}$$

$\downarrow$ apply x

For $f \in M^*$

$$(x.\overline{m})(f) \overset{?}{=} \overline{x.m}\,(f)$$

$$= f(x.m)$$

$$\overline{m}(x^S.f)$$

$$(x^S.f)(m)$$

$$f((x^S)^S.m)$$

Require

$$(x^S)^S = x \qquad \forall x \in A$$

ie

$$S^2 = I$$

So ☆ is not an A module hom for $A = U_q$

Next goal: show that in <u>some cases</u> we can modify ☆ so it becomes an A-module hom.

Assumption 3| Given Hopf alg $(A, \Delta, \varepsilon, S)$

Assume $\exists$ invertible $k \in A$ st

$$S^2(x) = k^{-1} x k \qquad \forall x \in A$$

Note that Assumption holds for $A = U_q$

2/13/9

LEM 32 Under Assumption 31 the map

$$M \longrightarrow (M^*)^*$$
$$m \longrightarrow \overline{k^{-1}.m}$$

$\bar{m}(f) = f(m)$

is an A-module hom

pf For $x \in A$

$$\begin{array}{ccc} M & \longrightarrow & (M^*)^* \\ m & \longrightarrow & \overline{k^{-1}.m} \\ \downarrow & & \downarrow \\ x.m & & x.\overline{k^{-1}.m} \\ & \searrow & \overline{k^{-1}.x.m} \end{array} \quad \overset{?}{=}$$

$\downarrow$ apply x

For $f \in M^*$

$$(x.\overline{k^{-1}.m})(f) \overset{?}{=} \overline{k^{-1}.x.m}(f)$$

$$= \overline{k^{-1}.m}(x^S.f)$$
$$= (x^S.f)(k^{-1}.m)$$
$$= f((x^S)^S.k^{-1}.m)$$
$$= f(k^{-1}xkk^{-1}.m)$$
$$= f(k^{-1}x.m)$$

$$\overline{k^{-1}.x.m}(f) = f(k^{-1}.x.m) = f(k^{-1}x.m)$$

OK

□

COR 33 For the Hopf alg

$$(U_q, \Delta, \varepsilon, S)$$

and a U_q-module M,

$$\begin{aligned} M &\longrightarrow (M^\vee)^* \\ m &\longrightarrow \overline{k^{-1}.m} \end{aligned}$$

is a U_q module hom.

Pf By LEM 32 and since

$$S^2(x) = k^{-1} x k \qquad \forall x \in U_q$$

$\square$

The quantum trace

We will define a quantum trace tr_q for our Hopf alg $(U_q, \Delta, \varepsilon, S)$

We start with some comments.

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Given A-module M

Recall

$$M^* \otimes M \longrightarrow \mathbb{F}$$

$$f \otimes m \longrightarrow f(m)$$

is an A-module hom

What about

$$M \otimes M^* \longrightarrow \mathbb{F} \qquad ?$$

$$m \otimes f \longrightarrow f(m)$$

Not A-module hom in gen; $A = U_q$ is counterexample

LEM 34 Under Assumption 31,

for an A-module M the map

$$\begin{aligned} M \otimes M^* &\to \mathbb{F} \\ m \otimes f &\mapsto f(k^{-1}.m) \end{aligned}$$

is an A-module hom.

pf This is the composition

$$\begin{aligned} M \otimes M^* &\longrightarrow (M^*)^* \otimes M^* \longrightarrow \mathbb{F} \\ m \otimes f &\longmapsto \left(\overline{k^{-1}.m}\right) \otimes f \end{aligned}$$

can

A-module hom

$\square$

COR 35 For our Hopf alg

$$(U_q, \Delta, \varepsilon, S)$$

and a U_q-module M the map

$$M \otimes M^* \to \mathbb{F}$$

$$m \otimes f \to f(k^{-1} m)$$

is a U_q-module hom. $\square$

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Given A-modules M, N

Recall

$$\mathrm{can}: \begin{array}{ccc} N \otimes M^* & \to & \mathrm{Hom}(M, N) \\ n \otimes f & \to & \varphi_{f,n} \end{array} \qquad \varphi_{f,n}(m) = f(m)n$$

is A-module hom.

<u>Special case</u> $N = M$

$$\mathrm{can}: \quad M \otimes M^* \to \mathrm{Hom}(M, M) = \mathrm{End}(M)$$

is A-module hom.

<u>More special case</u> $\dim M < \infty$

$\mathrm{can}: M \otimes M^* \to \mathrm{End}(M)$

is bijection and hence iso of A-modules