

A few comments

LEM 17 Over map $S: U_q \rightarrow U_q$ satisfies

$$(i) \quad S(e^n) = (-1)^n q^{n(n-1)/2} k^{-n} e^n \quad n \in \mathbb{N}$$

$$(ii) \quad S(f^n) = (-1)^n q^{n(n+1)/2} p^n k^n \quad --$$

$$(iii) \quad S(k^n) = k^{-n} \quad --$$

$$(iv) \quad S(k^{-n}) = k^n \quad --$$

□

pf Induction on n LEM 18 Over map $S: U_q \rightarrow U_q$ satisfies

$$(i) \quad S^{-1}(e) = -ek^{-1}$$

$$(ii) \quad S^{-1}(f) = -kf$$

$$(iii) \quad S^{-1}(k) = k^{-1}$$

$$(iv) \quad S^{-1}(k^{-1}) = k$$

pf Routine

□

DEF 19 For a U_q -module M ,

a bilinear form

$$\langle \cdot, \cdot \rangle : M \times M \rightarrow \mathbb{F}$$

is called U_q -invariant whenever

$$\langle x \cdot u, v \rangle = \langle u, S(x) \cdot v \rangle \quad \forall u, v \in M \quad \forall x \in U_q$$

(Caution: $\langle \cdot, \cdot \rangle$ not symmetric in gen)

Problem 20 Find a nondegenerate U_q -invariant bilinear form for the U_q -module $L(n, \varepsilon)$ and q not a root of 1.

Problem 21

L12/3

Recall that U_q has a basis

$$f^r k^a e^t \quad r, t \in \mathbb{N}, a \in \mathbb{Z}$$

Define a bilinear form

$$\langle \cdot, \cdot \rangle : U_q \times U_q \rightarrow \mathbb{F}$$

st

$$\langle f^r k^a e^t, f^R k^S e^T \rangle =$$

$$\delta_{r,R} \delta_{r,t} \frac{(-1)^{r+t} q^{\binom{r}{2}} [r]_q! q^{\binom{t}{2}} [t]_q! q^{2r - (2-r)(S-R)}}{(q - q^{-1})^{r+t}}$$

Show that

• $\langle \cdot, \cdot \rangle$ is U_q -invariant

• $\langle v, u \rangle = \langle \omega_S(u), \omega_S(v) \rangle \quad \forall u, v \in U_q$

• In the discussion

$$U_q^{(0)} = \sum_{n \in \mathbb{N}} f^n \Lambda e^n$$

The summands are mutually orthogonal

• Assume q not root of 1. Then for $n \in \mathbb{Z}$ the bil form $\langle \cdot, \cdot \rangle$ gives a nondegenerate pairing

$$\langle \cdot, \cdot \rangle \quad U_q^{(n)} \times U_q^{(-n)} \rightarrow \mathbb{F}$$

Hopf Algebras

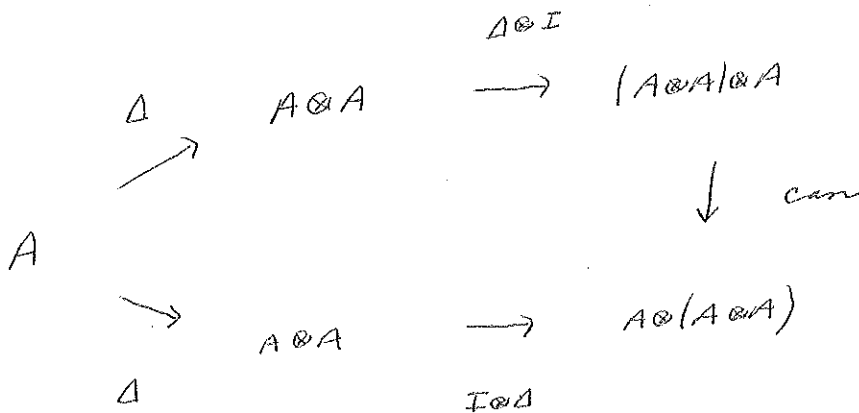
DEF 22 A Hopf algebra over \mathbb{F} is a sequence

$$(A, \Delta, \epsilon, S)$$

such that

- A is an algebra over \mathbb{F}
- $\Delta: A \rightarrow A \otimes A$ is an algebra hom (counit)
- $\epsilon: A \rightarrow \mathbb{F}$ (counit)
- $S: A \rightarrow A$ is an algebra antihom (antipode)

and the following diagrams commute



$$\begin{array}{ccc}
 & \Delta & \\
 A & \rightarrow & A \otimes A \\
 I \downarrow & & \downarrow \varepsilon \otimes I \\
 A & \rightarrow & F \otimes A \\
 \text{can} & &
 \end{array}$$

$$\begin{array}{ccc}
 & \Delta & \\
 A & \rightarrow & A \otimes A \\
 I \downarrow & & \downarrow I \otimes \varepsilon \\
 A & \rightarrow & A \otimes F \\
 \text{can} & &
 \end{array}$$

$$\begin{array}{ccc}
 & \Delta & \\
 A & \rightarrow & A \otimes A \\
 i \otimes \varepsilon \downarrow & & \downarrow S \otimes I \\
 A & \leftarrow & A \otimes A \\
 m & &
 \end{array}$$

$$\begin{array}{ccc}
 & \Delta & \\
 A & \rightarrow & A \otimes A \\
 i \otimes \varepsilon \downarrow & & \downarrow I \otimes S \\
 A & \leftarrow & A \otimes A \\
 m & &
 \end{array}$$

— 0 —

Obs $(U, \Delta, \varepsilon, S)$

U is a Hopf algebra.

Examples of Hopf algebras

Let $G = \text{group}$

Recall the group algebra $U = \mathbb{F}G$ has basis G

LEM 23 With the above notation,

(i) \exists alg hom $\Delta: U \rightarrow U \otimes U$ s.t.

$$\Delta(x) = x \otimes x \quad \forall x \in G$$

(ii) \exists alg hom $\varepsilon: U \rightarrow \mathbb{F}$ s.t.

$$\varepsilon(x) = 1 \quad \forall x \in G$$

(iii) \exists alg antihom $S: U \rightarrow U$ s.t.

$$S(x) = x^{-1} \quad \forall x \in G$$

(iv) $(U, \Delta, \varepsilon, S)$ is a Hopf algebra.

pf (ex)

(i) check $\Delta(x) \Delta(y) = \Delta(xy) \quad \forall x, y \in G$

$$\text{LHS} = (x \otimes x)(y \otimes y)$$

$$= xy \otimes xy$$

$$= \text{RHS} \quad \checkmark$$

(ii) check

$$\Sigma(x) \Sigma(y) = \Sigma(xy) \quad x, y \in G$$

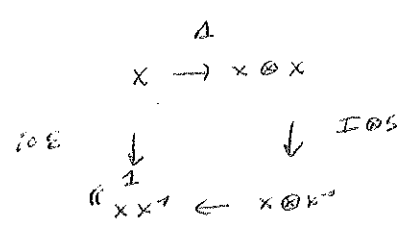
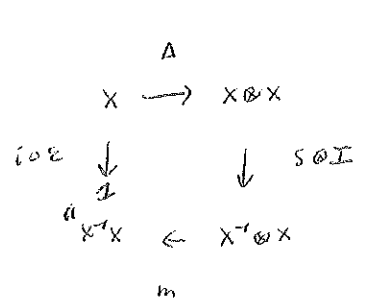
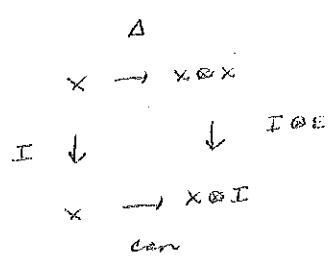
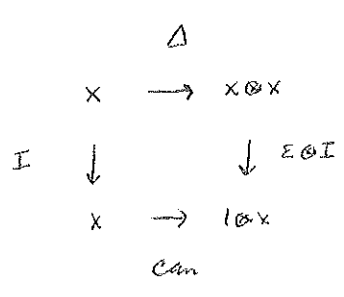
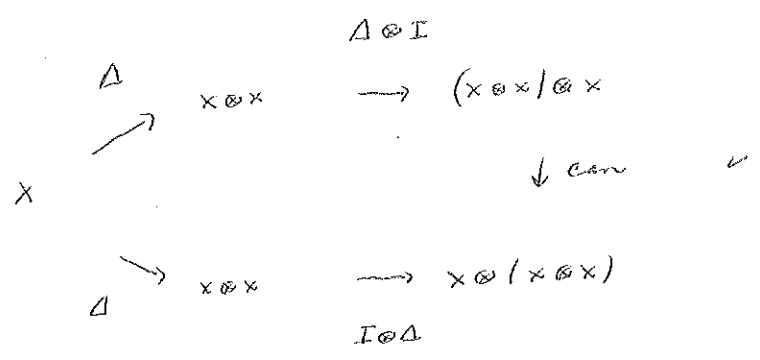
both sides 1 ✓

(iii) check

$$S(y) S(x) = S(xy) \quad x, y \in G$$

$$LHS = y^{-1} x^{-1} = (xy)^{-1} = RHS \quad \checkmark$$

(iv) show diag's commute: $\forall x \in G$



Let \mathfrak{g} = Lie algebra over \mathbb{F}

So \mathfrak{g} is a vector space with a map

$$[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

st

• $[\cdot, \cdot]$ is bilinear

$$• [x, x] = 0 \quad \forall x \in \mathfrak{g}$$

• $\forall x, y, z \in \mathfrak{g}$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

"Jacobi identity"

Recall the universal enveloping algebra $U = U(\mathfrak{g})$

is the (assoc) algebra with generators \mathfrak{g} and rules

$$xy - yx = [x, y] \quad \forall x, y \in \mathfrak{g}$$

Given a basis x_1, x_2, \dots for \mathfrak{g}

U has a basis

$$x_1^{r_1} x_2^{r_2} \dots$$

where r_1, r_2, \dots range over \mathbb{N}

LEM 24 With the above notation,

$$(i) \quad \exists \text{ alg hom } \Delta: U \rightarrow U \otimes U \quad \text{st} \\ \Delta(x) = x \otimes 1 + 1 \otimes x \quad \forall x \in \mathfrak{g}$$

$$(ii) \quad \exists \text{ alg hom } \varepsilon: U \rightarrow \mathbb{F} \quad \text{st} \\ \varepsilon(x) = 0 \quad \forall x \in \mathfrak{g}$$

$$(iii) \quad \exists \text{ alg antihom } S: U \rightarrow U \quad \text{st} \\ S(x) = -x \quad \forall x \in \mathfrak{g}$$

(iv) $(U, \Delta, \varepsilon, S)$ is a Hopf algebra.

pf (ex)

(i) Show Δ respects the defining rels for U :

$$\Delta(x) \Delta(y) - \Delta(y) \Delta(x) = \Delta([x, y]) \quad \forall x, y \in \mathfrak{g}$$

LHS =

$$\begin{aligned}
 & (x \otimes 1 + 1 \otimes x) (y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y) (x \otimes 1 + 1 \otimes x) \\
 = & x y \otimes 1 + x \otimes y + y \otimes x + 1 \otimes x y \\
 & - (y x \otimes 1 + x \otimes y + y \otimes x + 1 \otimes y x) \\
 = & (x y - y x) \otimes 1 + 1 \otimes (x y - y x) \\
 = & [x, y] \otimes 1 + 1 \otimes [x, y] \\
 = & \text{RHS}
 \end{aligned}$$

(ii) Check ε respects the defining rels for U :

$$\varepsilon(x) \varepsilon(y) - \varepsilon(y) \varepsilon(x) \stackrel{?}{=} \varepsilon([x, y])$$

 $x, y \in \mathfrak{g}$

both sides 0 ✓

(iii) Show S respects the defining rels for U :

$$S(y) S(x) - S(x) S(y) \stackrel{?}{=} S([x, y])$$

 $x, y \in \mathfrak{g}$

$$\text{LHS} = (-y)(-x) - (-x)(-y)$$

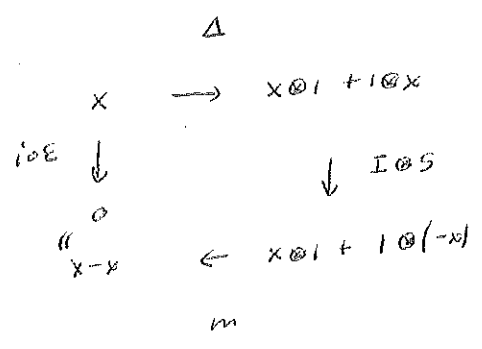
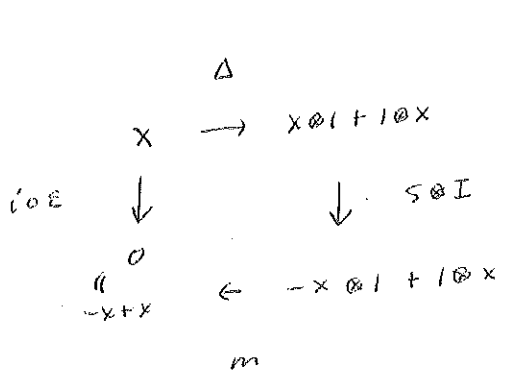
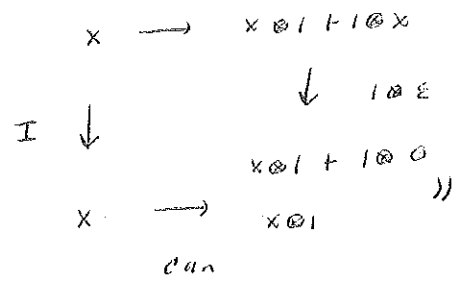
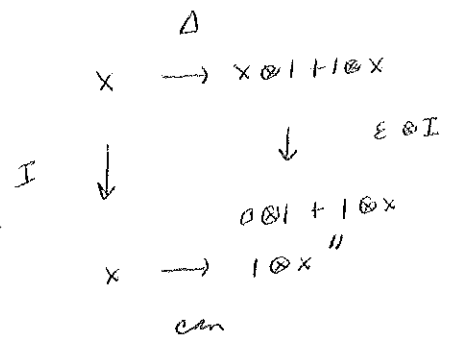
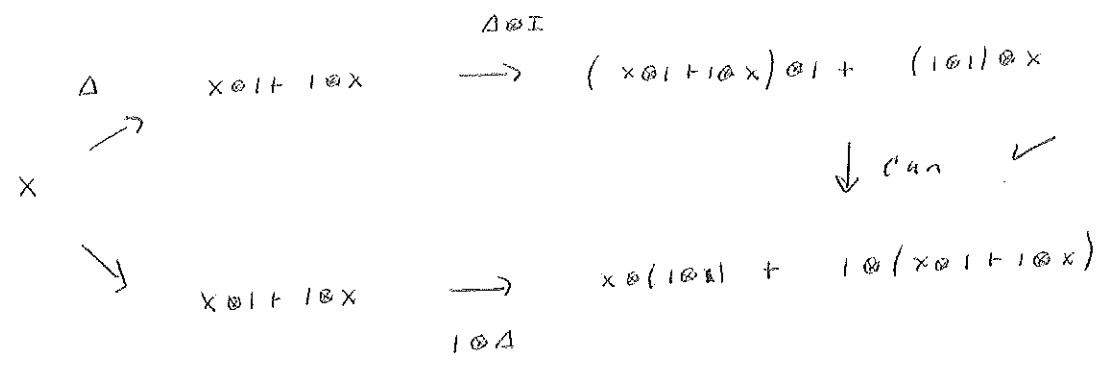
$$= -(xy - yx)$$

$$= -[x, y]$$

$$= \text{RHS} \quad \checkmark$$

(iv) Show diagrams commute. By LEM 14
 it suffices to check things for the gens of \mathcal{U}

$\forall x \in \mathfrak{g}$,



□

Another canonical map

Given vector spaces M, N

Recall $\text{Hom}(M, N) = \text{Hom}_{\mathbb{F}}(M, N)$

is the vector space of all \mathbb{F} -linear maps $M \rightarrow N$

So $\text{Hom}(M, M) = \text{End}(M)$

DEF 25 With above notation,

consider the \mathbb{F} -linear map

$$N \otimes M^{\vee} \rightarrow \text{Hom}(M, N) \quad *$$

$$n \otimes f \rightarrow \varphi_{f,n}$$

where

$$\varphi_{f,n}(m) = f(m)n \quad m \in M$$

We call $*$ the canonical map

We remark that $*$ is a bijection if M, N have finite dimensions.

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Assume M, N are A -modules

So M^* becomes A -module via S

$N \otimes M^*$ becomes A -module via Δ

Next goal: Find an A -module structure on

$\text{Hom}(M, N)$

is

can: $N \otimes M^* \rightarrow \text{Hom}(M, N)$

is a hom of A -modules

Strategy

- First view $N \otimes M^*$ as $(A \otimes A)$ -module
- Find $(A \otimes A)$ -module str on $\text{Hom}(M, N)$
- It can be seen that $\text{Hom}(M, N)$ is $(A \otimes A)$ -module
- Using Δ , the $(A \otimes A)$ -module $\text{Hom}(M, N)$ becomes the desired A -module

Given $x, y \in A$

$$N \otimes M^* \xrightarrow{\text{can}} \text{Hom}(M, N)$$

$$n \otimes f \xrightarrow{\quad} \varphi_{f, n}$$



$$x \otimes y \cdot (n \otimes f)$$

$$x \otimes y \cdot (\varphi_{f, n})$$

) desired

$$= (x \cdot n) \otimes (y \cdot f)$$



$$\varphi_{y \cdot f, x \cdot n}$$

\downarrow applies
 $x \otimes y$

$$\forall m \in M$$

$$(x \otimes y \circ (\varphi_{f,n})) (m) \stackrel{\text{desired}}{=} \varphi_{y,f, x \circ n} (m)$$

$$= (y \circ f)(m) (x \circ n)$$

$$= f(y^S, m) x \circ n$$

$$= x \circ (f(y^S, m) n)$$

Since $f(y^S, m) \in F$

$$= x \circ (\varphi_{f,n} (y^S, m))$$

So the desired $(A \otimes A)$ -module $\text{Hom}(M, N)$ should satisfy:

$$\forall x, y \in A \quad \forall \varphi \in \text{Hom}(M, N),$$

$$M \longrightarrow N$$

$$x \otimes y \circ \varphi : m \longrightarrow x \circ (\varphi (y^S, m))$$

First we check this really gives a $(A \otimes A)$ -module str.

LEM 2b For a Hopf alg $(A, \Delta, \varepsilon, S)$

For A -modules M, N .

$\text{Hom}(M, N)$ is an $(A \otimes A)$ -module. $\forall \varepsilon$

$\forall x, y \in A \quad \forall \varphi \in \text{Hom}(M, N)$,

$$\begin{array}{ccc}
 & M & \longrightarrow N \\
 x \otimes y \circ \varphi & m & \longrightarrow x \cdot (\varphi(y \cdot m))
 \end{array}$$

pt For $x, y, x', y' \in A$ and $\varphi \in \text{Hom}(M, N)$

$$x x' \otimes y y' \circ \varphi \stackrel{?}{=} x \otimes y \circ (x' \otimes y' \circ \varphi)$$

at $m \in M$ each side is

$$x x' \circ \varphi(S(y') S(x) m)$$

OK

□