Lecture 12

A few comments

LEM 17 Our map $S : U_q \to U_q$ satisfies

(i) $S(e^n) = (-1)^n q^{n(n-1)} k^{-n} e^n \qquad n \in \mathbb{N}$

(ii) $S(f^n) = (-1)^n q^{n(1-n)} f^n k^n \qquad$ "

(iii) $S(k^n) = k^{-n} \qquad$ "

(iv) $S(k^{-n}) = k^n \qquad$ "

$\square$

pf Induction on n

LEM 18 Our map $S : U_q \to U_q$ satisfies

(i) $S^{-1}(e) = -ek^{-1}$

(ii) $S^{-1}(f) = -kf$

(iii) $S^{-1}(k) = k^{-1}$

(iv) $S^{-1}(k^{-1}) = k$

pf Routine $\square$

DEF 19 For a U_q-module M,

a bilinear form

$$\langle\,,\,\rangle : M \times M \to \mathbb{F}$$

is called U_q-<u>invariant</u> whenever

$$\langle x.u, v\rangle = \langle u, S(x).v\rangle \quad \forall\, u,v \in M \quad \forall\, x \in U_q$$

(Caution: $\langle\,,\,\rangle$ not symmetric in gen)

Problem 20 Find a nondegenerate U_q-invariant bilinear form for the U_q-module $L(n,\varepsilon)$ and q not a root of 1.

Problem 21

Recall that U_q has a basis

$$f^r k^s e^t \qquad r,t \in \mathbb{N},\ s \in \mathbb{Z}$$

Define a bilinear form

$$\langle\ ,\ \rangle : U_q \times U_q \to \mathbb{F}$$

st

$$\langle f^r k^s e^t, f^R k^S e^T \rangle =$$

$$\delta_{r,T}\,\delta_{R,t}\,\frac{(-1)^{r+t}\, q^{\binom{r}{2}}\,[r]_q^!\; q^{\binom{t}{2}}\,[t]_q^!\; q^{2r-(s-r)(S-R)}}{(q-q^{-1})^{r+t}}$$

Show that

- $\langle , \rangle$ is U_q-invariant
- $\langle v, u \rangle = \langle \omega S(u), \omega S(v) \rangle \quad \forall u, v \in U_q$
- In the dir sum

$$U_q^{(0)} = \sum_{n \in \mathbb{N}} f^n \Lambda e^n$$

the summands are mut orthogonal

- Assume q not root of 1. Then for $n \in \mathbb{Z}$ the bil form $\langle , \rangle$ gives a nondegenerate pairing

$$\langle , \rangle \quad U_q^{(n)} \times U_q^{(-n)} \to \mathbb{F}$$

Hopf Algebras

DEF 22 A <u>Hopf algebra</u> over $\mathbb{F}$ is a sequence

$$(A, \Delta, \varepsilon, S)$$

such that

- A is an algebra over $\mathbb{F}$ (comult)
- $\Delta : A \to A \otimes A$ is an algebra hom (counit)
- $\varepsilon : A \to \mathbb{F}$ ... (antipode)
- $S : A \to A$ is an algebra antihom

and the following diagrams commute

$$\begin{tikzcd}
 & A \otimes A \arrow[r, "\Delta \otimes I"] & (A \otimes A) \otimes A \arrow[d, "can"] \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \\
 & A \otimes A \arrow[r, "I \otimes \Delta"'] & A \otimes (A \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "I"'] & A \otimes A \arrow[d, "\varepsilon \otimes I"] \\ A \arrow[r, "\text{can}"'] & \mathbb{F} \otimes A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "I"'] & A \otimes A \arrow[d, "I \otimes \varepsilon"] \\ A \arrow[r, "\text{can}"'] & A \otimes \mathbb{F} \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "i \circ \varepsilon"'] & A \otimes A \arrow[d, "S \otimes I"] \\ A & A \otimes A \arrow[l, "m"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "i \circ \varepsilon"'] & A \otimes A \arrow[d, "I \otimes S"] \\ A & A \otimes A \arrow[l, "m"] \end{tikzcd}$$

— o —

Obs $(U_q, \Delta, \varepsilon, S)$

is a Hopf algebra.

Examples of Hopf algebras

Let G = group

Recall the group algebra $U = \mathbb{F}G$ has basis G

LEM 23 With the above notation,

(i) $\exists$ alg hom $\Delta : U \to U \otimes U$ s.t

$$\Delta(x) = x \otimes x \qquad \forall x \in G$$

(ii) $\exists$ alg hom $\varepsilon : U \to \mathbb{F}$ s.t

$$\varepsilon(x) = 1 \qquad \forall x \in G$$

(iii) $\exists$ alg antihom $S : U \to U$ s.t

$$S(x) = x^{-1} \qquad \forall x \in G$$

(iv) $(U, \Delta, \varepsilon, S)$ is a Hopf algebra.

pf (ex)

(i) Check $\Delta(x)\,\Delta(y) = \Delta(xy) \qquad \forall x, y \in G$

$$\text{LHS} = (x \otimes x)(y \otimes y)$$

$$= xy \otimes xy$$

$$= \text{RHS} \quad ✓$$

(ii) check

$$\varepsilon(x)\,\varepsilon(y) = \varepsilon(xy) \qquad x, y \in G$$

both sides 1 ✓

(iii) check

$$S(y)\,S(x) = S(xy) \qquad x, y \in G$$

$$\text{LHS} = y^{-1}x^{-1} = (xy)^{-1} = \text{RHS} \checkmark$$

(iv) Show diags commute: $\forall x \in G$

$$\begin{tikzcd} & x\otimes x \arrow[r, "\Delta\otimes I"] & (x\otimes x)\otimes x \arrow[d, "\text{can}"] \\ x \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \\ & x\otimes x \arrow[r, "I\otimes\Delta"'] & x\otimes(x\otimes x) \end{tikzcd}$$

✓

$$\begin{tikzcd} x \arrow[r, "\Delta"] \arrow[d, "I"'] & x\otimes x \arrow[d, "\varepsilon\otimes I"] \\ x \arrow[r, "\text{can}"'] & 1\otimes x \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\Delta"] \arrow[d, "I"'] & x\otimes x \arrow[d, "I\otimes\varepsilon"] \\ x \arrow[r, "\text{can}"'] & x\otimes 1 \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, "\Delta"] \arrow[d, "i\circ\varepsilon"'] & x\otimes x \arrow[d, "S\otimes I"] \\ 1 = x^{-1}x & x^{-1}\otimes x \arrow[l, "m"] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\Delta"] \arrow[d, "i\circ\varepsilon"'] & x\otimes x \arrow[d, "I\otimes S"] \\ 1 = xx^{-1} & x\otimes x^{-1} \arrow[l] \end{tikzcd}$$

$\square$

let $\mathfrak{g}$ = Lie algebra over $\mathbb{F}$

So $\mathfrak{g}$ is a vector space with a map

$$[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

st

- $[\,,\,]$ is bilinear
- $[x,x] = 0 \quad \forall x \in \mathfrak{g}$
- For $x,y,z \in \mathfrak{g}$

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$$

"Jacobi identity"

Recall the universal enveloping algebra $U = U(\mathfrak{g})$

is the (assoc) algebra with generators $\mathfrak{g}$ and rels

$$xy - yx = [x,y] \qquad \forall x,y \in \mathfrak{g}$$

Given a basis $x_1, x_2, \ldots$ for $\mathfrak{g}$

U has a basis

$$x_1^{r_1} x_2^{r_2} \cdots$$

where $r_1, r_2, \ldots$ range over $\mathbb{N}$

LEM 24 With the above notation,

(i) $\exists$ alg hom $\Delta : U \to U \otimes U$ st

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad \forall x \in \mathfrak{g}$$

(ii) $\exists$ alg hom $\varepsilon : U \to \mathbb{F}$ s.t

$$\varepsilon(x) = 0 \qquad \forall x \in \mathfrak{g}$$

(iii) $\exists$ alg antihom $S : U \to U$ st

$$S(x) = -x \qquad \forall x \in \mathfrak{g}$$

(iv) $(U, \Delta, \varepsilon, S)$ is a Hopf algebra.

pf (ex)

(i) Show Δ respects the defining rels for U:

$$\Delta(x)\Delta(y) - \Delta(y)\Delta(x) \overset{?}{=} \Delta\left([x,y]\right) \qquad \forall x,y \in \mathfrak{g}$$

LHS =

$$(x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y) - (y\otimes 1 + 1\otimes y)(x\otimes 1 + 1\otimes x)$$

$$= xy\otimes 1 + x\otimes y + y\otimes x + 1\otimes xy$$
$$- (yx\otimes 1 + x\otimes y + y\otimes x + 1\otimes yx)$$

$$= (xy - yx)\otimes 1 + 1\otimes (xy - yx)$$

$$= [x,y]\otimes 1 + 1\otimes [x,y]$$

= RHS

(ii) Check ε respects the defining rels for U: $\quad x,y\in\mathfrak{g}$

$$\varepsilon(x)\varepsilon(y) - \varepsilon(y)\varepsilon(x) \stackrel{?}{=} \varepsilon([x,y])$$

both sides 0 ✓

(iii) Show S respects the defining rels for U: $\quad x,y\in\mathfrak{g}$

$$S(y)S(x) - S(x)S(y) \stackrel{?}{=} S([x,y])$$

$$\text{LHS} = (-y)(-x) - (-x)(-y)$$
$$= -(xy - yx)$$
$$= -[x,y]$$
$$= \text{RHS} \quad ✓$$

(iv) Show diagrams commute. By LEM 14 it suffices to check things for the gens of U

$\forall x \in \mathfrak{g}$,

$$\begin{tikzcd}
 & x\otimes 1 + 1\otimes x \arrow[r, "\Delta\otimes I"] & (x\otimes 1 + 1\otimes x)\otimes 1 + (1\otimes 1)\otimes x \arrow[d, "can\ \checkmark"] \\
x \arrow[ur, "\Delta"] \arrow[dr] & & \\
 & x\otimes 1 + 1\otimes x \arrow[r, "1\otimes\Delta"'] & x\otimes(1\otimes 1) + 1\otimes(x\otimes 1 + 1\otimes x)
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "\Delta"] \arrow[dd, "I"'] & x\otimes 1 + 1\otimes x \arrow[d, "\varepsilon\otimes I"] \\
 & 0\otimes 1 + 1\otimes x \arrow[d, equal] \\
x \arrow[r, "can"'] & 1\otimes x
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r] \arrow[dd, "I"'] & x\otimes 1 + 1\otimes x \arrow[d, "1\otimes\varepsilon"] \\
 & x\otimes 1 + 1\otimes 0 \arrow[d, equal] \\
x \arrow[r, "can"'] & x\otimes 1
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "\Delta"] \arrow[d, "i\circ\varepsilon"'] & x\otimes 1 + 1\otimes x \arrow[d, "S\otimes I"] \\
0 = -x+x & -x\otimes 1 + 1\otimes x \arrow[l, "m"]
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, "\Delta"] \arrow[d, "i\circ\varepsilon"'] & x\otimes 1 + 1\otimes x \arrow[d, "I\otimes S"] \\
0 = x-x & x\otimes 1 + 1\otimes(-x) \arrow[l, "m"]
\end{tikzcd}$$

$\square$

Another canonical map

Given vector spaces M, N

Recall $\quad Hom(M,N) = Hom_{\mathbb{F}}(M,N)$

is the vector space of all $\mathbb{F}$-linear maps $M \to N$

So $\quad Hom(M,M) = End(M)$

DEF 25 $\quad$ With above notation,

consider the $\mathbb{F}$-linear map

$$N \otimes M^{\vee} \longrightarrow Hom(M,N) \qquad *$$

$$n \otimes f \longrightarrow \varphi_{f,n}$$

where

$$\varphi_{f,n}(m) = f(m)\, n \qquad m \in M$$

We call $*$ the <u>canonical map</u>

We remark that $*$ is a bijection if M, N have finite dimensions

Given Hopf alg $(A, \Delta, \varepsilon, S)$

Assume M, N are A-modules

So M^* becomes A-module via S

$N \otimes M^*$ becomes A-module via Δ

<u>Next goal</u> : Find an A-module structure on

$\mathrm{Hom}(M,N)$

s.t.

$\mathrm{can} : N \otimes M^* \longrightarrow \mathrm{Hom}(M,N)$

is a hom of A-modules

Strategy

- First view $N \otimes M^*$ as $(A \otimes A)$-module
- Find $(A \otimes A)$-module str on $\mathrm{Hom}(M,N)$ st can is hom of $(A \otimes A)$-modules
- Using Δ, the $(A \otimes A)$-module $\mathrm{Hom}(M,N)$ becomes the desired A-module

Given $x, y \in A$

$$\begin{tikzcd}
N \otimes M^* \arrow[r, "\mathrm{can}"] & \mathrm{Hom}(M,N) \\
n \otimes f \arrow[r, mapsto] \arrow[d, "\text{apply } x \otimes y"'] & \varphi_{f,n} \arrow[d] \\
 & x \otimes y \cdot (\varphi_{f,n}) \arrow[d, phantom, "\text{desire}\ \|"] \\
x \otimes y \cdot (n \otimes f) = (x.n) \otimes (y.f) \arrow[r, mapsto] & \varphi_{y.f,\, x.n}
\end{tikzcd}$$

$\forall m \in M$

$$\left(x \otimes y \cdot (\varphi_{f,n}) \right)(m) \overset{\text{desire}}{=} \varphi_{y \cdot f,\, x \cdot n}(m)$$

$$= (y \cdot f)(m)\,(x \cdot n)$$

$$= f(y^S \cdot m)\; x \cdot n$$

$$= x \cdot \left(f(y^S \cdot m)\; n \right) \qquad \text{since } f(y^S \cdot m) \in F$$

$$= x \cdot \left(\varphi_{f,n}(y^S \cdot m) \right)$$

So the desired $(A \otimes A)$-module $Hom(M, N)$ should satisfy:

$\forall x, y \in A \qquad \forall \varphi \in Hom(M, N),$

$$x \otimes y \cdot \varphi : \begin{array}{ccc} M & \longrightarrow & N \\ m & \longmapsto & x \cdot \left(\varphi(y^S \cdot m) \right) \end{array}$$

First we check this really gives a $(A \otimes A)$-module str.

LEM 26 For a Hopf alg $(A, \Delta, \varepsilon, S)$

For A-modules M, N.

$Hom(M,N)$ is an $(A \otimes A)$-module st

$\forall x, y \in A$ $\forall \varphi \in Hom(M,N)$,

$$x \otimes y \cdot \varphi \qquad \begin{array}{ccc} M & \longrightarrow & N \\ m & \longrightarrow & x \cdot (\varphi(y^S \cdot m)) \end{array}$$

pf For $x, y, x', y' \in A$ and $\varphi \in Hom(M,N)$

$$xx' \otimes yy' \cdot \varphi \overset{?}{=} x \otimes y \cdot (x' \otimes y' \cdot \varphi)$$

at $m \in M$ each side is

$$xx' \cdot \varphi(S(y')S(y)\, m)$$

ok

$\square$