

Lecture 11

LEM 9 For our maps

$$\Delta: U_1 \rightarrow U_1 \otimes U_1$$

$$\varepsilon: U_1 \rightarrow \mathbb{F}$$

the following diagrams commute:

$$\begin{array}{ccc} U_1 & \xrightarrow{\Delta} & U_1 \otimes U_1 \\ I \downarrow & & \downarrow \varepsilon \otimes I \\ U_1 & \xrightarrow{\text{can}} & \mathbb{F} \otimes U_1 \\ & & \text{can} \end{array}$$

$$\begin{array}{ccc} U_1 & \xrightarrow{\Delta} & U_1 \otimes U_1 \\ I \downarrow & & \downarrow I \otimes \varepsilon \\ U_1 & \xrightarrow{\text{can}} & U_1 \otimes \mathbb{F} \\ & & \text{can} \end{array}$$

pf

Recall

$$\varepsilon(e) = 0, \quad \varepsilon(f) = 0, \quad \varepsilon(k) = 1, \quad \varepsilon(k^{-1}) = 1$$

To show diag on left commutes, we verify:

$$\forall x \in U_1 \quad x = \sum_i \varepsilon(x_i) x_i' \quad \text{where} \quad \Delta(x) = \sum_i x_i \otimes x_i'$$

Suf to verify for

$$x \in \{e, f, k, k^{-1}\}$$

Case $x = e$:

$$\text{Recall } \Delta(e) = e \otimes 1 + k \otimes e$$

$$e \stackrel{?}{=} \underbrace{\varepsilon(e)}_0 \cdot 1 + \underbrace{\varepsilon(k)}_1 \cdot e \quad \text{OK}$$

other cases sim.

Diagram right is sim.



COR 10 For a U_q module M ,

(i) can: $M \rightarrow F \otimes M$ is iso of U_q modules

(ii) can: $M \rightarrow M \otimes F$ is iso of U_q modules

pf By the motivation discussion. □

The dual space

Let M denote a vector space

Recall the dual space M^* is the vector space

consisting of all the \mathbb{F} -linear maps $M \rightarrow \mathbb{F}$

Recall $\dim M = \dim M^*$ if $\dim M < \infty$

LEM 11 Given alg A
Given A module M

Given anti-autom S of A

Then M^* becomes an A -module as follows:

$$\forall x \in A \quad \forall f \in M^*$$

$x \cdot f$ is that element of M^* s.t.

$$(x \cdot f)(m) = f(x^S \cdot m) \quad \forall m \in M$$

$$[x^S \text{ means } S(x)]$$

pf Routine verification.

□

Cautin The anti-ant T is not the best
 choice for S in LEM 11.
 We now explain why.

DEF 12 Given vector space M

Consider \mathbb{F} -linear map

$$\begin{aligned} M^* \otimes M &\longrightarrow \mathbb{F} \\ f \otimes m &\longrightarrow f(m) \end{aligned}$$

Call this the canonical map

Motivation

Given A algebra
 alg hom $\Delta: A \rightarrow A \otimes A$
 alg hom $\epsilon: A \rightarrow \mathbb{F}$
 anti aut $S: A \rightarrow A$

A module M

view M^* as A module via S
 $M^* \otimes M$ as A module via Δ
 \mathbb{F} as A module via ϵ

We desire:

can: $M^* \otimes M \rightarrow \mathbb{F}$ is hom of A modules

Pick $x \in A$

write $\Delta(x) = \sum_i x_i \otimes x_i'$

$$\begin{array}{ccc}
 M^* \otimes M & \xrightarrow{\text{can}} & \mathbb{F} \\
 f \otimes m & \rightarrow & f(m)
 \end{array}$$

\downarrow apply x

$$\begin{array}{ccc}
 \downarrow & & \downarrow \\
 x_0(f \otimes m) & & \epsilon(x)f(m) \quad \rightsquigarrow \text{desire} \\
 = \sum_i (x_i \circ f) \otimes (x_i' \circ m) & \rightarrow & \sum_i (x_i \circ f)(x_i' \circ m)
 \end{array}$$

$$\begin{aligned}
 \mathbb{E}(x) f(m) & \stackrel{\text{desire}}{=} \sum_i (x_{i \circ f})(x'_{i \circ m}) \\
 & = \sum_i f(x_i^S(x'_{i \circ m})) \\
 & = \sum_i f((x_i^S x'_i)_{\circ m}) \\
 & = f\left(\left(\sum_i x_i^S x'_i\right)_{\circ m}\right)
 \end{aligned}$$

The desired condition is

$$\mathbb{E}(x) 1 = \sum_i x_i^S x'_i$$

This condition is expressed by saying the following diagram commutes:

$$\begin{array}{ccc}
 & \Delta & \\
 A & \longrightarrow & A \otimes A \\
 \downarrow \scriptstyle i \circ \mathbb{E} & & \downarrow \scriptstyle S \otimes I \\
 A & \xleftarrow{m} & A \otimes A
 \end{array}$$

Here

$$\begin{aligned}
 m: A \otimes A & \rightarrow A \\
 a \otimes a' & \rightarrow aa'
 \end{aligned}$$

mult map

$$\begin{aligned}
 i: F & \rightarrow A \\
 d & \rightarrow d1
 \end{aligned}$$

inclusion map

Next goal:

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For $A = Uz$, find S that makes above diag commute

Require of S :

$$\begin{array}{ccc}
 & & A \\
 & & \longrightarrow \\
 e & & e \otimes 1 + k \otimes e \\
 \downarrow \varepsilon & & \downarrow \\
 0 & & s(e) \otimes 1 + s(k) \otimes e \\
 \text{require } \left(\begin{array}{c} 0 \\ s(e) + s(k)e \end{array} \right) & \longleftarrow & \\
 & & m
 \end{array}$$

$$\begin{array}{ccc}
 & & f \\
 & & \longrightarrow \\
 f & & f \otimes k^{-1} + 1 \otimes f \\
 \downarrow & & \downarrow \\
 0 & & s(f) \otimes k^{-1} + 1 \otimes f \\
 \text{require } \left(\begin{array}{c} 0 \\ s(f)k^{-1} + f \end{array} \right) & \longleftarrow &
 \end{array}$$

$$\begin{array}{ccc}
 & & k \\
 & & \longrightarrow \\
 k & & k \otimes k \\
 \downarrow & & \downarrow \\
 1 & & s(k) \otimes k \\
 \text{require } \left(\begin{array}{c} 1 \\ s(k)k \end{array} \right) & \longleftarrow &
 \end{array}$$

So

$$s(k) = k^{-1}$$

$$s(e) = -k^{-1}e$$

$$s(f) = -fk$$

Still need to check this really is an anticommutator

LEM B \exists unique anti-ant S of U_q st

(ii) $S(e) = -k^{-1}e$

(iii) $S(f) = -fk$

(iv) $S(k) = k^{-1}$

(v) $S(k^{-1}) = k$

Moreover

$$S^2(x) = k^{-1}xk$$

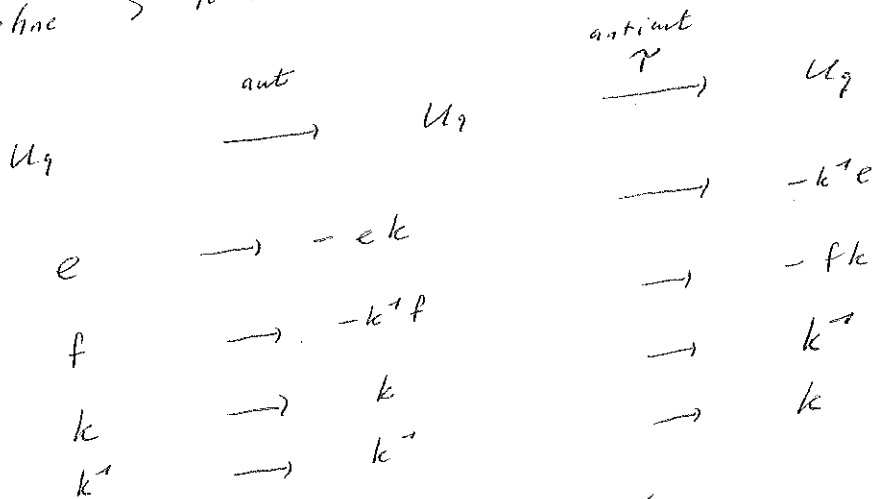
$$\forall x \in U_q$$

(*)

Call S the antipode of U_q

pf existence:

Define S to be the composition



S is anti-ant of U_q that satis (i)-(iv)

Uniqueness: e, f, k, k^{-1} gen U_q ✓

(*) : verify for $x \in \{e, f, k, k^{-1}\}$

□

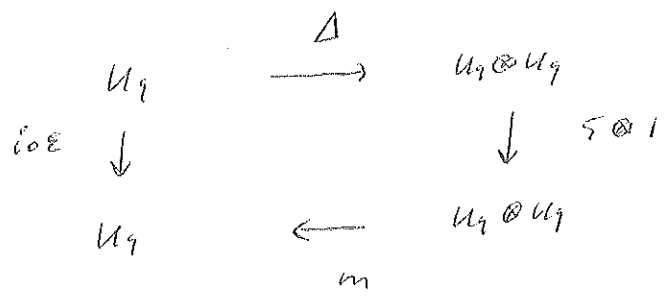
next goal:

L11/9

For our maps

$$\Delta: U_2 \rightarrow U_2 \otimes U_2, \quad \varepsilon: U_2 \rightarrow \mathbb{F}, \quad S: U_2 \rightarrow U_2$$

wish to verify the following diagram commutes:



Since m is not an algebra hom, A priori it is not enough to verify things for just the gens e, t, k, k^2 of U_2

Let us examine this issue.

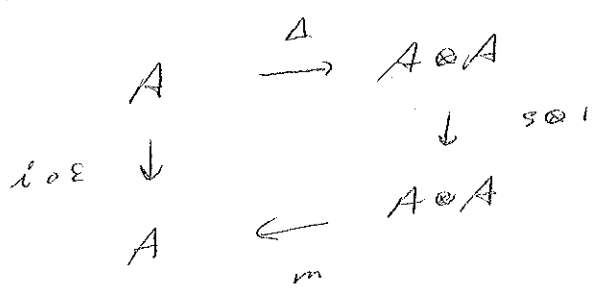
Given alg A

algebra $\Delta: A \rightarrow A \otimes A$

$\varepsilon: A \rightarrow \mathbb{F}$

antiauto $S: A \rightarrow A$

wish to determine if this commutes:



Must show

LL1/10

$$\forall x \in A$$

$$\varepsilon(x) = \sum_i s(x_i) x_i'$$

$$\text{where } \Delta(x) = \sum_i x_i \otimes x_i' \quad *$$

LEM 14 let $G \subseteq A$ denote a generating set for A

Assume $(*)$ holds for $x \in G$.

then $(*)$ holds for $x \in A$

pf Define

$$A' = \{ x \in A \mid (*) \text{ holds for } x \}$$

show $A' = A$

By assumption

$$G \subseteq A'$$

show A' is subalg of A

Given $x, y \in A'$ show $xy \in A'$

By constr

$$\varepsilon(x) = \sum_i s(x_i) x_i'$$

$$\varepsilon(y) = \sum_i s(y_i) y_i'$$

$$\Delta(x) = \sum_i x_i \otimes x_i'$$

$$\Delta(y) = \sum_i y_i \otimes y_i'$$

obs

$$\begin{aligned} \Delta(xy) &= \Delta(x) \Delta(y) \\ &= \sum_i \sum_j x_i y_j \otimes x_i' y_j' \end{aligned}$$

We need

$$\mathbb{E}(xy) \stackrel{?}{=} \sum_i \sum_j S(x_i y_j) x_i' y_j'$$

$$\begin{aligned} \text{RHS} &= \sum_i \sum_j S(y_j) S(x_i) x_i' y_j' \\ &= \sum_j S(y_j) \left(\sum_i S(x_i) x_i' \right) y_j' \\ &= \sum_j S(y_j) \mathbb{E}(x) y_j' \\ &= \mathbb{E}(x) \sum_j S(y_j) y_j' \\ &= \mathbb{E}(x) \mathbb{E}(y) \\ &= \mathbb{E}(xy) \\ &= \text{LHS} \quad \checkmark \end{aligned}$$

□

COR 15 For our maps

$$\Delta: U_1 \rightarrow U_1 \otimes U_1, \quad \varepsilon: U_1 \rightarrow \mathbb{F}, \quad S: U_1 \rightarrow U_1$$

The following diagram commutes:

$$\begin{array}{ccc}
 & \Delta & \\
 U_1 & \longrightarrow & U_1 \otimes U_1 \\
 \downarrow \text{id} & & \downarrow S \otimes I \\
 U_1 & \xlongequal{m} & U_1 \otimes U_1
 \end{array}$$

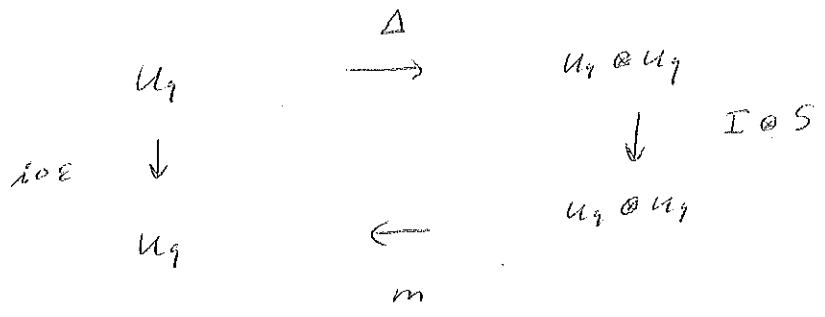
pf By LEM 14 it suffices to verify the diag
 $\text{for } e, f, k, k^{-1}$. The verification is routine. \square

A "bonus"

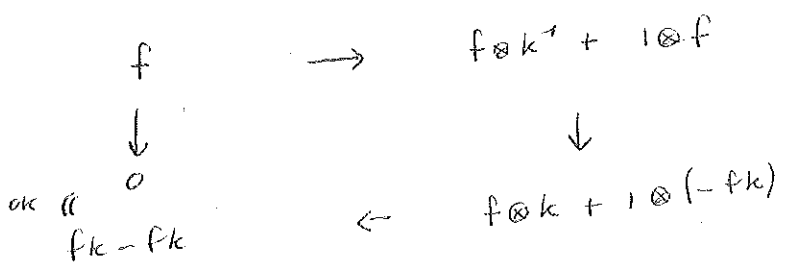
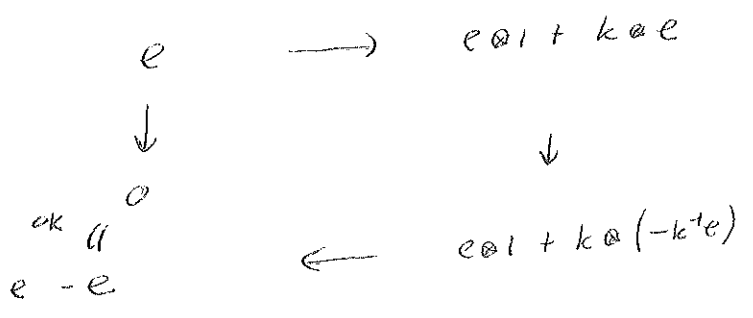
LEM 16 For our maps

$$\Delta: U_2 \rightarrow U_2 \otimes U_2, \quad \varepsilon: U_2 \rightarrow \mathbb{F}, \quad S: U_2 \rightarrow U_2$$

The following diag commutes:



pt As in Cor 15, it suffices to verify the diag for e, f, k, k^{-1} .



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$$\begin{array}{ccc} k & \longrightarrow & k \otimes k \\ \downarrow & & \downarrow \\ \text{det} \left(\begin{array}{c} | \\ k \otimes k^{-1} \end{array} \right) & \longleftarrow & k \otimes k^{-1} \end{array}$$

$$\begin{array}{ccc} k^{-1} & \longrightarrow & k^{-1} \otimes k^{-1} \\ \downarrow & & \downarrow \\ \text{det} \left(\begin{array}{c} | \\ k^{-1} \otimes k \end{array} \right) & \longleftarrow & k^{-1} \otimes k \end{array}$$

□