Lecture 11

LEM 9 For our maps

$\Delta : U_q \to U_q \otimes U_q$ $\qquad \varepsilon : U_q \to \mathbb{F}$

the following diagrams commute:

$$\begin{tikzcd} U_q \arrow[r, "\Delta"] \arrow[d, "I"'] & U_q \otimes U_q \arrow[d, "\varepsilon \otimes I"] \\ U_q \arrow[r, "\mathrm{can}"'] & \mathbb{F} \otimes U_q \end{tikzcd} \qquad \begin{tikzcd} U_q \arrow[r, "\Delta"] \arrow[d, "I"'] & U_q \otimes U_q \arrow[d, "I \otimes \varepsilon"] \\ U_q \arrow[r, "\mathrm{can}"'] & U_q \otimes \mathbb{F} \end{tikzcd}$$

pf Recall

$\varepsilon(e) = 0, \quad \varepsilon(f) = 0, \quad \varepsilon(k) = 1, \quad \varepsilon(k^{-1}) = 1$

To show diag on left commutes, we verify:

$\forall\, x \in U_q$

$$x = \sum_i \varepsilon(x_i)\, x_i' \quad \text{where} \quad \Delta(x) = \sum_i x_i \otimes x_i'$$

Suf to verify for

$$x \in \{e, f, k, k^{-1}\}$$

Case $x = e$:

Recall $\Delta(e) = e \otimes 1 + k \otimes e$

$$e \overset{?}{=} \underbrace{\varepsilon(e)}_{=0} 1 + \underbrace{\varepsilon(k)}_{=1} e \qquad \text{ok}$$

other cases sim.

Diag on right is sim. $\square$

COR 10 For a U_q-module M,

(i) can: $M \to \mathbb{F} \otimes M$ is iso of U_q-modules

(ii) can: $M \to M \otimes \mathbb{F}$ is iso of U_q-modules

pf By the motivation discussion. $\square$

The dual space

Let M denote a vector space

Recall the dual space M^* is the vector space consisting of all the $\mathbb{F}$-linear maps $M \to \mathbb{F}$

Recall $\dim M = \dim M^*$ if $\dim M < \infty$

LEM 11 Given alg A

Given A module M

Given antiaut S of A

Then M^* becomes an A module as follows:

$\forall x \in A \quad \forall f \in M^*$

$x.f$ is that element of M^* s.t

$$(x.f)(m) = f(x^S . m) \qquad \forall m \in M$$

[x^S means $S(x)$]

pf Routine verification. $\square$

<u>Caution</u> The antiaut T is not the best choice for S in LEM 11.

We now explain why.

DEF 12 Given vector space M

Consider $\mathbb{F}$-linear map

$$M^* \otimes M \longrightarrow \mathbb{F}$$
$$f \otimes m \mapsto f(m)$$

Call this the <u>canonical map</u>

Motivation

Given alg A

alg hom $\Delta : A \to A \otimes A$

alg hom $\varepsilon : A \to \mathbb{F}$

antiaut $S : A \to A$

A module M

View M^* as A module via S

$M^* \otimes M$ as A module via Δ

$\mathbb{F}$ as A module via ε

We desire:

can: $M^* \otimes M \longrightarrow \mathbb{F}$ is hom of A modules

Pick $x \in A$

write $\Delta(x) = \sum_i x_i \otimes x_i'$

$$\begin{array}{lccc}
 & M^* \otimes M & \xrightarrow{\text{can}} & \mathbb{F} \\
 & f \otimes m & \longrightarrow & f(m) \\
\downarrow^{\text{apply } x} & \downarrow & & \downarrow \\
 & x.(f \otimes m) & & \varepsilon(x) f(m) \quad \urcorner\urcorner \text{ desire} \\
 & = \sum_i (x_i . f) \otimes (x_i' . m) & \longrightarrow & \sum_i (x_i . f)(x_i' . m)
\end{array}$$

$$\varepsilon(x) f(m) \overset{\text{desire}}{=} \sum_i (x_i \circ f)(x_i' \circ m)$$

$$= \sum_i f(x_i^S \circ (x_i' \circ m))$$

$$= \sum_i f((x_i^S x_i') \circ m)$$

$$= f\left(\left(\sum_i x_i^S x_i'\right) \circ m\right)$$

The desired condition is

$$\varepsilon(x) 1 = \sum_i x_i^S x_i'$$

This condition is expressed by saying the following diag commutes:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "i \circ \varepsilon"'] & A \otimes A \arrow[d, "S \otimes I"] \\ A & A \otimes A \arrow[l, "m"] \end{tikzcd}$$

Here

$$m : \begin{array}{l} A \otimes A \to A \\ a \otimes a' \to a a' \end{array} \qquad \text{mult map}$$

$$i : \begin{array}{l} \mathbb{F} \to A \\ \alpha \to \alpha 1 \end{array} \qquad \text{inclusion map}$$

Next goal:

For $A = U_q$, find S that makes above diag commute

Require of S:

$$\begin{tikzcd}
e \arrow[r, "\Delta"] \arrow[d, "\varepsilon"'] & e\otimes 1 + k\otimes e \arrow[d, "S\otimes I"] \\
0 & S(e)\otimes 1 + S(k)\otimes e \arrow[l, "m"]
\end{tikzcd}$$

require $0 = S(e) + S(k)e$

$$\begin{tikzcd}
f \arrow[r] \arrow[d] & f\otimes k^{-1} + 1\otimes f \arrow[d] \\
0 & S(f)\otimes k^{-1} + 1\otimes f \arrow[l]
\end{tikzcd}$$

require $0 = S(f)k^{-1} + f$

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k\otimes k \arrow[d] \\
1 & S(k)\otimes k \arrow[l]
\end{tikzcd}$$

require $1 = S(k)k$

So

$$S(k) = k^{-1}$$
$$S(e) = -k^{-1}e$$
$$S(f) = -fk$$

Still need to check this really is an antipode.

LEM B $\exists$ unique antiaut S of U_q st

(i) $S(e) = -k^{-1}e$

(ii) $S(f) = -fk$

(iii) $S(k) = k^{-1}$

(iv) $S(k^{-1}) = k$

Moreover

$$S^2(x) = k^{-1}xk \qquad \forall x \in U_q \qquad (*)$$

Call S the <u>antipode</u> of U_q

pf existence:

Define S to be the composition

$$U_q \xrightarrow{\text{aut}} U_q \xrightarrow[\text{antiaut}]{\tau} U_q.$$

$$\begin{array}{ccccc}
e & \longrightarrow & -ek & \longrightarrow & -k^{-1}e \\
f & \longrightarrow & -k^{-1}f & \longrightarrow & -fk \\
k & \longrightarrow & k & \longrightarrow & k^{-1} \\
k^{-1} & \longrightarrow & k^{-1} & \longrightarrow & k
\end{array}$$

S is antiaut of U_q that sats (i)–(iv)

Uniqueness: e, f, k, k^{-1} gen U_q ✓

$(*)$: verify for $x \in \{e, f, k, k^{-1}\}$ $\square$

Next goal:

For our maps

$\Delta: U_q \to U_q \otimes U_q$, $\quad \varepsilon: U_q \to \mathbb{F}$, $\quad S: U_q \to U_q$

wish to verify the following diagram commutes:

$$\begin{tikzcd} U_q \arrow[r, "\Delta"] \arrow[d, "i\circ\varepsilon"'] & U_q \otimes U_q \arrow[d, "S\otimes 1"] \\ U_q & U_q \otimes U_q \arrow[l, "m"] \end{tikzcd}$$

Since m is not an alg hom, Apriori it is not enough to verify things for just the gens e, f, k, k^{-1} of U_q

Let us examine this issue.

Given alg A

alg hom $\Delta: A \to A \otimes A$

" $\varepsilon: A \to \mathbb{F}$

antiaut $S: A \to A$

Wish to determine if this commutes:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "i\circ\varepsilon"'] & A \otimes A \arrow[d, "S\otimes 1"] \\ A & A \otimes A \arrow[l, "m"] \end{tikzcd}$$

Must show

$\forall x \in A$

$$\varepsilon(x) = \sum_i S(x_i) x_i' \quad \text{where } \Delta(x) = \sum_i x_i \otimes x_i' \quad *$$

LEM 14 Let $G \subseteq A$ denote a generating set for A

Assume $(*)$ holds for $x \in G$.

Then $(*)$ holds for $x \in A$

pf Define

$$A' = \{ x \in A \mid * \text{ holds for } x \}$$

show $A' = A$

By assumption $G \subseteq A'$

show A' is subalg of A

Given $x, y \in A'$ show $xy \in A'$

By constr

$$\varepsilon(x) = \sum_i S(x_i) x_i' \qquad \Delta(x) = \sum x_i \otimes x_i'$$

$$\varepsilon(y) = \sum_i S(y_i) y_i' \qquad \Delta(y) = \sum y_i \otimes y_i'$$

Obs

$$\Delta(xy) = \Delta(x)\Delta(y)$$

$$= \sum_i \sum_j x_i y_j \otimes x_i' y_j'$$

We need

$$\varepsilon(xy) \overset{?}{=} \sum_i \sum_j S(x_i y_j)\, x_i' y_j'$$

$$RHS = \sum_i \sum_j S(y_j) S(x_i)\, x_i' y_j'$$

$$= \sum_j S(y_j) \Big(\sum_i S(x_i)\, x_i' \Big) y_j'$$

$$= \sum_j S(y_j) \quad \varepsilon(x)\, y_j'$$

$$= \varepsilon(x) \sum_j S(y_j)\, y_j'$$

$$= \varepsilon(x)\ \varepsilon(y)$$

$$= \varepsilon(xy)$$

$$= LHS \quad \checkmark$$

$\square$

COR 15 For our maps

$\Delta: U_q \to U_q \otimes U_q$, $\varepsilon: U_q \to \mathbb{F}$, $S: U_q \to U_q$

the following diag commutes:

$$\begin{tikzcd} U_q \arrow[r, "\Delta"] \arrow[d, "i \circ \varepsilon"'] & U_q \otimes U_q \arrow[d, "S \otimes I"] \\ U_q & U_q \otimes U_q \arrow[l, "m"] \end{tikzcd}$$

pf By LEM 14 it suffices to verify the diag for e, f, k, k^{-1}. The verification is routine. $\square$

A "bonus"

LEM 16 For our maps

$\Delta : U_q \to U_q \otimes U_q$, $\quad \varepsilon : U_q \to \mathbb{F}$, $\quad S : U_q \to U_q$

the following diag commutes:

$$\begin{tikzcd}
U_q \arrow[r, "\Delta"] \arrow[d, "i \circ \varepsilon"'] & U_q \otimes U_q \arrow[d, "I \otimes S"] \\
U_q & U_q \otimes U_q \arrow[l, "m"]
\end{tikzcd}$$

pf As in COR 15, it suffices to verify the diag for e, f, k, k^{-1}.

$$\begin{tikzcd}
e \arrow[r] \arrow[d] & e \otimes 1 + k \otimes e \arrow[d] \\
\begin{matrix} 0 \\ \text{ok } \| \\ e - e \end{matrix} & e \otimes 1 + k \otimes (-k^{-1} e) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r] \arrow[d] & f \otimes k^{-1} + 1 \otimes f \arrow[d] \\
\begin{matrix} 0 \\ \text{ok } \| \\ fk - fk \end{matrix} & f \otimes k + 1 \otimes (-fk) \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd} k \arrow[r] \arrow[d] & k \otimes k \arrow[d] \\ \text{ok} \quad 1 = k k^{-1} & k \otimes k^{-1} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} k^{-1} \arrow[r] \arrow[d] & k^{-1} \otimes k^{-1} \arrow[d] \\ \text{ok} \quad 1 = k^{-1} k & k^{-1} \otimes k \arrow[l] \end{tikzcd}$$

$\square$