Lecture 10

Motivation

Let M = any non 0 vs over $\mathbb{F}$

Let A = any $\mathbb{F}$-alg

An A-module str on M is equiv to an $\mathbb{F}$-alg hom $A \to End(M)$

Given A-modules M, N

Consider tensor product

$$M \otimes N = M \otimes_{\mathbb{F}} N$$

(so if m_i is basis for M and n_j is basis for N then $m_i \otimes n_j$ is basis for $M \otimes N$)

Wish to define an A-module str on $M \otimes N$

Proceed as follows.

Observe that

$$A \otimes A$$

is an $\mathbb{F}$-algebra with mult

$$(a_1 \otimes a_1')(a_2 \otimes a_2') = (a_1 a_2) \otimes (a_1' a_2')$$

Obs that $M \otimes N$ is an $A \otimes A$-module with action

$$(a \otimes a') \cdot (m \otimes n) = (a \cdot m) \otimes (a' \cdot n)$$

This gives $\mathbb{F}$-alg hom

$$A \otimes A \longrightarrow \mathrm{End}(M \otimes N) \qquad *$$

Suppose we can find an $\mathbb{F}$-alg hom

$$A \to A \otimes A \qquad **$$

Composing $*$, $**$ we get an $\mathbb{F}$-alg hom

$$A \to \mathrm{End}(M \otimes N)$$

Using this we get the desired A-module str on $M \otimes N$ via our prelim comments.

LEM 1 $\quad \exists$ unique $\mathbb{F}$-alg hom

$$\Delta : U_q \to U_q \otimes U_q$$

st

$$\Delta(e) = e \otimes 1 + k \otimes e$$

$$\Delta(f) = f \otimes k^{-1} + 1 \otimes f$$

$$\Delta(k) = k \otimes k$$

$$\Delta(k^{-1}) = k^{-1} \otimes k^{-1}$$

We call Δ the "comultiplication" for U_q

pf $\quad$ Require

(i) $\quad \Delta(k)\Delta(k^{-1}) = \Delta(k^{-1})\Delta(k) = 1$

(ii) $\quad \Delta(k)\Delta(e) = q^2 \Delta(e)\Delta(k)$

(iii) $\quad \Delta(k)\Delta(f) = q^{-2}\Delta(f)\Delta(k)$

(iv) $\quad \Delta(e)\Delta(f) - \Delta(f)\Delta(e) = \dfrac{\Delta(k) - \Delta(k^{-1})}{q - q^{-1}}$

check (i) ✓

Check (ii)

$$LHS = (k \otimes k)(e \otimes 1 + k \otimes e)$$

$$= ke \otimes k + k^2 \otimes ke$$

$$= q^2 ek \otimes k + k^2 \otimes q^2 ek$$

$$= q^2 (e \otimes 1 + k \otimes e)(k \otimes k)$$

$$= RHS$$

Check (iii), (iv) sim. $\square$

COR 2 Let M, N denote U_q-modules.

Then $M \otimes N$ is a U_q-module with the following action:

$\forall x \in U_q \quad \forall m \in M \quad \forall n \in N$

$$x \cdot (m \otimes n) = \sum_i (x_i \cdot m) \otimes (x_i' \cdot n)$$

where

$$\Delta(x) = \sum_i x_i \otimes x_i'$$

In particular

$$e \cdot (m \otimes n) = (e \cdot m) \otimes n + (k \cdot m) \otimes (e \cdot n)$$

$$f \cdot (m \otimes n) = (f \cdot m) \otimes (k^{-1} \cdot n) + m \otimes (f \cdot n)$$

$$k \cdot (m \otimes n) = (k \cdot m) \otimes (k \cdot n)$$

$$k^{-1} \cdot (m \otimes n) = (k^{-1} \cdot m) \otimes (k^{-1} \cdot n)$$

$\square$

pf By the motivation discussion.

Given vector spaces M_1, M_2, M_3 over $\mathbb{F}$

Recall

$$(M_1 \otimes M_2) \otimes M_3 \qquad *$$

iso

$$M_1 \otimes (M_2 \otimes M_3) \qquad **$$

An iso is

$$(m_1 \otimes m_2) \otimes m_3 \longrightarrow m_1 \otimes (m_2 \otimes m_3)$$

Call this the <u>Canonical iso</u>

Next assume M_1, M_2, M_3 are U_q-modules.

By our previous comments each of $*$, $**$ has a U_q-module str. We want the canon map to be an iso of U_q-modules from $*$ to $**$.

This will occur if the map $\Delta : U_q \to U_q \otimes U_q$ has the following property.

DEF 3 For an alg A,

an algebra hom

$$\Delta : A \to A \otimes A$$

is <u>co-associative</u> whenever the following diagram

commutes:

$$\begin{tikzcd}
 & A \otimes A \arrow[r, "I \otimes \Delta"'] & A \otimes (A \otimes A) \arrow[d, "\mathrm{can}"] \\
A \arrow[ur, "\Delta"'] \arrow[dr, "\Delta"'] & & \\
 & A \otimes A \arrow[r, "\Delta \otimes I"'] & (A \otimes A) \otimes A
\end{tikzcd}$$

LEM 4 Our map $\Delta : U_q \to U_q \otimes U_q$ is coassoc

pf We show the diag commutes:

$$\begin{tikzcd}
 & U_q \otimes U_q \arrow[r, "1 \otimes \Delta"] & U_q \otimes (U_q \otimes U_q) \arrow[d, "\text{can}"] \\
U_q \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \\
 & U_q \otimes U_q \arrow[r, "\Delta \otimes I"'] & (U_q \otimes U_q) \otimes U_q
\end{tikzcd}$$

For $x \in U_q$ we apply the various maps and check the results agree. Since each map is an alg hom, suf to check for $x \in \{e, f, k, k^{-1}\}$

$x = e$:

$$\begin{tikzcd}
 & e \otimes 1 + k \otimes e \arrow[r] & e \otimes (1 \otimes 1) + k \otimes (e \otimes 1 + k \otimes e) = e \otimes (1 \otimes 1) + k \otimes (e \otimes 1) + k \otimes (k \otimes e) \arrow[d] \\
e \arrow[ur] \arrow[dr] & & (e \otimes 1) \otimes 1 + (k \otimes e) \otimes 1 + (k \otimes k) \otimes e \\
 & e \otimes 1 + k \otimes e \arrow[r] & (e \otimes 1 + k \otimes e) \otimes 1 + (k \otimes k) \otimes e \arrow[u, equal]
\end{tikzcd}$$

case of f, k, k^{-1} sim $\square$

Comment

LEM 5 Our map $\Delta : U_q \to U_q \otimes U_q$ satisfies

(i) $$\Delta(e^n) = \sum_{r=0}^{n} q^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_q e^{n-r} k^r \otimes e^r$$

(ii) $$\Delta(f^n) = \sum_{r=0}^{n} q^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_q f^r \otimes f^{n-r} k^{-r}$$

$\forall n \in \mathbb{N}$

pf Routine induction on n. $\square$

the "trivial" U_q-module

LEM 6 $\exists$ alg hom $\varepsilon : U_q \to \mathbb{F}$ that sends

$$e \to 0, \quad f \to 0, \quad k \to 1, \quad k^{-1} \to 1$$

We call ε the <u>counit</u> of U_q.

pf Routine $\square$

DEF 7 Using ε above we view $\mathbb{F}$ as a 1 dim'l U_q-module as follows:

$\forall x \in U_q \quad \forall v \in \mathbb{F}$

$$x . v = \varepsilon(x) v$$

We obs that $e, f, k-1, k^{-1}-1$ vanish on this module.

We call $\mathbb{F}$ the <u>trivial</u> U_q module

Def 8 View field $\mathbb{F}$ as 1 dim'l vector space / $\mathbb{F}$

Let M denote any vector space / $\mathbb{F}$.

Then

$$M \to M \otimes \mathbb{F} \qquad *$$

$$m \to m \otimes 1$$

is an iso of VS. Also

$$M \longrightarrow \mathbb{F} \otimes M \qquad **$$

$$m \longrightarrow 1 \otimes m$$

is an iso of VS.

Call $*$, $**$ the <u>Canonical</u> isomorphisms

L10|12

Let M denote a U_q module

View $\mathbb{F}$ as U_q module as in Def 7

Using $\Delta : U_q \to U_q \otimes U_q$ we get a U_q module str

on $\mathbb{F} \otimes M$ and $M \otimes \mathbb{F}$.

We desire

can: $M \to M \otimes \mathbb{F}$ is iso of U_q modules

can: $M \longrightarrow \mathbb{F} \otimes M$ is iso of U_q modules

In order to motivate things we first consider a more general situation.

The trivial module — Motivation

Given algebra A

Given alg hom $\Delta : A \to A \otimes A$

Given alg hom $\varepsilon : A \to \mathbb{F}$

Given A module M

View $\mathbb{F}$ as A module via ε

View $\mathbb{F} \otimes M$ as A module via Δ

<u>We desire</u>:

$\mathrm{can} : M \to \mathbb{F} \otimes M$ is iso of A modules

Pick $x \in A$

Write $\Delta x = \sum_i x_i \otimes x_i'$ $\quad x_i, x_i' \in A$

apply x ↓

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{can}} & \mathbb{F} \otimes M \\ m & \longrightarrow & 1 \otimes m \\ \downarrow & & \downarrow \\ & & x \cdot (1 \otimes m) \\ x \cdot m & \longrightarrow & \quad \Vert \text{ desire} \\ & & 1 \otimes (x \cdot m) \end{array}$$

$$1 \otimes (x \cdot m) \overset{\text{desire}}{=} x \cdot (1 \otimes m)$$

$$= \Delta(x)\,(1 \otimes m)$$

$$= \sum_i (x_i \cdot 1) \otimes (x_i' \cdot m)$$

$$= \sum_i \varepsilon(x_i)\, 1 \otimes (x_i' \cdot m)$$

$$= 1 \otimes \Big(\sum_i \varepsilon(x_i)\, x_i'\Big) \cdot m$$

The desired condition is

$$x = \sum_i \varepsilon(x_i)\, x_i'$$

This condition is expressed by saying that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "I"'] & A \otimes A \arrow[d, "\varepsilon \otimes I"] \\
A \arrow[r, "can"'] & \mathbb{F} \otimes A
\end{tikzcd}$$

Similarly,

$\mathrm{can}\quad M \longrightarrow M \otimes \mathbb{F}$ is iso of A-modules

provided the following diag commutes:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "I"'] & A \otimes A \arrow[d, "I \otimes \varepsilon"] \\ A \arrow[r, "\mathrm{can}"'] & A \otimes \mathbb{F} \end{tikzcd}$$