University of Wisconsin-Madison
Math 846: Introduction to quantum groups
Lecture 1, MWF 11:00–11:50, Van Vleck B235
Syllabus for Semester II, 2018/2019

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Text: Lectures on quantum groups, by Jens Carsten Jantzen, Graduate Studies in Mathematics Vol. 6 ISBN-13: 978-0821804780 ISBN-10: 0821804782

Prerequisites: Good understanding of linear algebra.

Course Content: In this introductory course we will discuss the basic concepts associated with quantum groups.

We will begin with a concrete example: the quantum group $U_q(\mathfrak{sl}_2)$. We will define this algebra via generators and relations; we will obtain a basis; we will compute the center, and we will describe the finite dimensional modules. We will discuss how $U_q(\mathfrak{sl}_2)$ is a quantized enveloping algebra for the Lie algebra $\mathfrak{sl}_2$. We will discuss how $U_q(\mathfrak{sl}_2)$ has the structure of a Hopf algebra.

With the example of $U_q(\mathfrak{sl}_2)$ in mind, we will turn our attention to the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra. We will develop the theory of $U_q(\mathfrak{g})$ from first principles. Along the way we will encounter the following topics: The quantum trace; the Yang-Baxter equation; the triangular decomposition of $U_q(\mathfrak{g})$; modules for $U_q(\mathfrak{g})$; the center of $U_q(\mathfrak{g})$; the Harish-Chandra homomorphism; the Hopf algebra structure for $U_q(\mathfrak{g})$; R-matrices; a bilinear form which pairs the positive and negative parts of $U_q(\mathfrak{g})$; the braid group action and PBW type basis; crystal bases.

The lectures will be self contained and no prior knowledge of the subject is assumed. I will follow the text more or less, This course should be valuable to anyone interested in Lie theory, quantum groups, algebraic combinatorics, number theory, knot invariants, and statistical mechanical models.

Course Credits: 3. Each week there will be three 50 minute lectures.

Evaluation: There are no exams. Near the end of the semester each non-dissertator student is expected to give one lecture, on a topic either from the text or a related topic of your choice. As the time approaches I will organize the speaking schedule and suggest topics.

Course goals/Learning outcomes: Master the material presented in lecture. For this I recommend the following study strategy. After each lecture do the following: for each stated definition write out numerous examples and non examples. For each stated result, write your own proof starting from first principles and without looking at your notes. It is not important if your proof matches mine or not. Done properly this strategy is easy to carry out, since every result in the course builds naturally on what came before.

Lecture 1

Math 846

Introduction to Quantum groups

Paul Terwilliger

Text: Lectures on Quantum Groups
by Jens Jantzen

Our object of study is $U_q(\mathfrak{g})$

$\mathfrak{g}$ = finite-dim'l semi-simple Lie algebra over a field $\mathbb{F}$

$q \in \mathbb{F}$, $q \neq 0$, $q \neq 1$, $q \neq -1$

$U_q(\mathfrak{g})$ = "quantized" enveloping algebra of $\mathfrak{g}$

We start with special case $\mathfrak{g} = \mathfrak{sl}_2$

CH I The quantized enveloping algebra $U_q(sl_2)$

Throughout,

$\mathbb{F}$ = arbitrary field

$\mathbb{N} = \{0, 1, 2, \ldots\}$ natural numbers

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ integers

Fix $0 \neq q \in \mathbb{F}$ satisfies $q^2 \neq 1$

Motivation

Recall the Lie algebra $sl_2(\mathbb{F})$ char $\mathbb{F} \neq 2$

$$sl_2(\mathbb{F}) = \left\{ x \in Mat_2(\mathbb{F}) \;/\; \text{trace}(x) = 0 \right\}$$

has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$[h, e] = 2e \qquad\qquad [r, s] = rs - sr$$

$$[h, f] = -2f$$

$$[e, f] = h$$

By def the universal enveloping algebra $U(sl_2)$ is assoc $\mathbb{F}$-algebra with 1, defined by generators E, F, H and rels

$$HE - EH = 2E$$
$$HF - FH = -2F$$
$$EF - FE = H$$

In order to motivate a "quantized" version of $U(sl_2)$, consider the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad k = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

"$k = q^H$"

We find k^{-1} exists and

$$ke = q^2 ek$$
$$kf = q^{-2} fk$$
$$ef - fe = \frac{k - k^{-1}}{q - q^{-1}}$$

From now on,

- all vector spaces discussed are over $\mathbb{F}$
- all algebras discussed are assoc, over $\mathbb{F}$, have 1
- a subalgebra has same 1 as the parent algebra

Notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{Z}$$

DEF 1 The algebra $U_q(sl_2)$ is defined by gens

$$e, f, k, k^{-1}$$

and relations

(i) $kk^{-1} = k^{-1}k = 1$

(ii) $ke = q^2 ek$

(iii) $kf = q^{-2} fk$

(iv) $ef - fe = \dfrac{k - k^{-1}}{q - q^{-1}}$

Write $U_q = U_q(sl_2)$

Next goal: display a basis for U_q

LEM 2 For $n \geq 1$, the following holds in U_q:

$$ef^n = f^n e + [n]_q f^{n-1} \frac{Kq^{1-n} - K^{-1}q^{n-1}}{q - q^{-1}}$$

pf Routine induction on n □

LEM 3 $\quad U_q$ is spanned by

$$f^r k^s e^t \qquad s \in \mathbb{Z}, \quad r, t \in \mathbb{N} \qquad *$$

Moreover for $s \in \mathbb{Z}$ and $r, t \in \mathbb{N}$ and writing

$$x_{rst} = f^r k^s e^t,$$

(i) $\quad e \, x_{rst} = q^{-2s} x_{r,s,t+1} + [r]_q \dfrac{q^{1-r} x_{r-1,s+1,t} - q^{r-1} x_{r-1,s-1,t}}{q - q^{-1}}$

(ii) $\quad f \, x_{rst} = x_{r+1,s,t}$

(iii) $\quad k \, x_{rst} = q^{-2r} x_{r,s+1,t}$

(iv) $\quad k^{-1} x_{rst} = q^{2r} x_{r,s-1,t}$

pf (i) – (iv) are by DEF 1, LEM 2

let V = subspace of U_q spanned by $*$

Show $V = U_q$

V contains $1 = x_{000}$

By (i) – (iv)

$$eV \subseteq V, \quad fV \subseteq V, \quad k^{\pm 1} V \subseteq V$$

$e, f, k^{\pm 1}$ generate U_q so $\quad U_q V \subseteq V$

V is a left ideal of U_q that contains 1, so $V = U_q$

$\square$

<u>Recall</u> For an alg A

an <u>A-module</u> is a vector space V together

with a map

$$A \times V \to V$$
$$a \quad v \to a.v$$

such that

$$(a+b).v = a.v + b.v \qquad a, b \in A$$
$$(\alpha a).v = \alpha(a.v) \qquad u, v \in V$$
$$a.(u+v) = a.u + a.v \qquad \alpha \in \mathbb{F}$$
$$a.(\alpha u) = \alpha(a.u)$$
$$1.v = v$$
$$(ab).v = a.(b.v)$$

— o —

We are going to construct a U_q-module.

For any non 0 vector space V define

$\mathrm{End}(V)$ = algebra of all linear trans $V \to V$

Consider a vector space V with basis

$$x_{rst} \qquad s \in \mathbb{Z} \qquad r, t \in \mathbb{N}$$

[Here x_{rst} is an indeterminate, our point of view here is different from LEM 3]

LEM 4 The above vector space V becomes a U_q-module on which e, f, k, k^{-1} act as in LEM 3 (i)–(iv).

pf Define $E, F, K, K^{-1} \in End(V)$ that act on the x_{rst} as in LEM 3 (i)–(iv).

show E, F, K, K^{-1} satisfy the defining relations of U_q from DEF 1.

For example, check

$$KF = q^{-2} FK$$

Apply each side to a basis vector x_{rst}.

$$K F x_{rst} = K\, x_{r+1, s+1, t}$$

$$= q^{-2r-2}\, x_{r+1, s+1, t}$$

$$F K x_{rst} = q^{-2r} F\, x_{r, s+1, t}$$

$$= q^{-2r}\, x_{r+1, s+1, t}$$

So $\quad KF = q^{-2} FK \quad$ at x_{rst}

The other rels are similarly checked. $\square$

Thm 5 The vectors below form a basis for the vector space U_q:

$$f^r k^s e^t \qquad s \in \mathbb{Z} \qquad r, t \in \mathbb{N} \qquad *$$

pf $*$ spans U_q by Lem 3

show $*$ lin indep

Recall the U_q-module V from Lem 4

$\exists$ lin trans

$$U_q \longrightarrow V$$

$$u \longrightarrow u \cdot x_{000}$$

For $s \in \mathbb{Z}$ and $r, t \in \mathbb{N}$,

$$f^r k^s e^t \longrightarrow f^r k^s e^t \cdot x_{000} = \underbrace{f^r \cdot \Big(\underbrace{k^s \cdot \big(\underbrace{e^t \cdot x_{000}}_{= x_{00t}}\big)}_{= x_{0st}}\Big)}_{= x_{rst}}$$

The x_{rst} are lin indep so $*$ are lin indep $\square$

Next goal:

Given two basis elements for U_q,
express their product as a linear comb of basis elements.

This is routine once we:
find $e^r f^s$ as a linear comb of basis elements for U_q.

Notation

$$[n]_q^! = [n]_q [n-1]_q \cdots [2]_q [1]_q \qquad n \in \mathbb{N}$$

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q^!}{[r]_q^! \, [n-r]_q^!} \qquad 0 \le r \le n$$

$$[k;n]_q = \frac{kq^n - k^{-1}q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{Z}$$

Obs for $n \in \mathbb{Z}$

$$[k;n]_q e = \frac{kq^n - k^{-1}q^{-n}}{q-q^{-1}} e$$

$$= \frac{q^2 e k q^n - q^{-2} e k^{-1} q^{-n}}{q-q^{-1}}$$

$$= e[k;n+2]_q$$

Sim $\quad [k;n]_q f = f[k,n-2]_q$

THM 6 For $r, s \in \mathbb{N}$

$$e^r f^s =$$

$$\sum_{n=0}^{\min(r,s)} [n]_q! \begin{bmatrix} r \\ n \end{bmatrix}_q \begin{bmatrix} s \\ n \end{bmatrix}_q f^{s-n} [K; 1+n-r-s]_q [K; 2+n-r-s]_q \cdots [K; 2n-r-s]_q e^{r-n} \qquad *$$

pf Obs

$$[n]_q! \begin{bmatrix} r \\ n \end{bmatrix}_q \begin{bmatrix} s \\ n \end{bmatrix}_q = \begin{bmatrix} r \\ n \end{bmatrix}_q [s]_q [s-1]_q \cdots [s-n+1]_q$$

RHS makes sense for $0 \le n \le r$, and is zero for $n > s$.

So $*$ can be expressed as

$$\sum_{n=0}^{r} \begin{bmatrix} r \\ n \end{bmatrix}_q [s]_q \cdots [s-n+1]_q f^{s-n} [K; 1+n-r-s]_q \cdots [K; 2n-r-s]_q e^{r-n} \qquad **$$

show $**$ by ind on r

$r = 0$ ✓

$r = 1$ This is LEM 2

$r \ge 2$ View

$$e^r f^s = e(e^{r-1} f^s)$$

expand $e^{r-1} f^s$ using $**$ and induction,

and in the result eval $e f^i$ using LEM 2.

After simplifying we obtain $**$.

$\square$