Lecture 7 Wednesday Feb 3

8.2 Principal Ideal Domains

Let R denote a commutative ring.

DEF 1 R is called a Principal Ideal Domain (PID) whenever

(i) R is an integral domain,

(ii) each ideal in R is principal

Ex Any Euclidean domain is a PID

Assume R is a PID

Given nonzero $a, b \in R$

Ideal $Ra + Rb$ is principal

Write $Ra + Rb = Rd$ $\quad d \in R$

By Prop 10 in 8.1
d is a GCD of a, b

By Prop 12 in 8.1
d is unique up to mult by a unit in R.

<u>Recall</u> For an ideal I of R

I called <u>maximal</u> whenever

(i) $I \neq R$

(ii) there does not exist an ideal J of R s.t.

$$I \subsetneq J \subsetneq R$$

<u>Recall</u> For an ideal $I \subseteq R$,

I is <u>prime</u> whenever

(i) $I \neq R$,

(ii) For $a, b \in R$

$ab \in I$ implies $a \in I$ or $b \in I$

<u>Recall</u> Assume R has a $1 \neq 0$

For an ideal $I \subseteq R$.

I max'l $\iff$ R/I is field $\Rightarrow$ R/I is int. domain $\iff$ I prime

LEM 2 Assume R is a PID

Given a non0 prime ideal P of R

Then P is maximal.

pf Given an ideal I of R s.t

$$P \subsetneq I$$

Show $I = R$

Write $I = Rx$ $\quad x \notin P$,

Also $P = Rp$ $\quad 0 \neq p \in I$

Since $p \in I$, $p = xy$ $\quad y \in R$

Since P prime, $x \in P$ or $y \in P$
No yes

$$y = zp \quad z \in R$$

So

$$p = xy = xzp$$

$$(1 - xz)\underbrace{p}_{\neq 0} = 0$$

$$1 - xz = 0$$

x is unit

$$I = Rx = R$$

$\square$

LEM 3 . Given x = indet

Consider poly ring $R[x]$

TFAE $\quad // \mathcal{R}$

(i) $R[x]$ is a PID

(ii) R is a field

pf (i) → (ii) The map

$$\mathcal{R} \longrightarrow R \qquad *$$
$$f \longrightarrow f(0)$$

is a surjective ring homomorphism with ker $\mathcal{R}x$

$*$ induces ring iso

$$R \simeq \mathcal{R}/\mathcal{R}x$$

$\mathcal{R}$ is integral domain

→ R is ...

→ $\mathcal{R}/\mathcal{R}x$ is ...

→ $\mathcal{R}x$ is prime

→ $\mathcal{R}x$ is maxl

→ $\mathcal{R}/\mathcal{R}x$ is field

→ R is field

(ii) → (i) $\mathcal{R}$ is Eucl domain and hence a PID. □

DEF 4 Assume R is an integral domain.

Given a norm $N: R \to \mathbb{N}$.

N is called <u>Dedekind - Hasse</u> whenever

(i) N is positive,

(ii) For all nonzero $a, b \in R$, either $a \in Rb$ or $\exists\, c \in Ra + Rb$ s.t.

$$0 < N(c) < N(b)$$

Prop 5 Assume R is an integral domain.

TFAE

(i) R is a PID

(ii) R has a Dedekind – Hasse norm.

Pf (i) → (ii) Show R does not contain ideals $\{I_i\}_{i=1}^{\infty}$ such that $I_i \subsetneq I_{i+1}$ for $1 \le i < \infty$ (*)

Suppose such I_i exist. Define

$$I = \bigcup_{i=1}^{\infty} I_i$$

I is an ideal of R and

$$I_i \subsetneq I \qquad (1 \le i < \infty)$$

I is principal

$$I = Rd \qquad d \in R$$

$\exists\, i \quad (1 \le i < \infty)$ s.t

$$d \in I_i$$

Now $\qquad I_i = I \qquad$ cont

* proved.

For $0 \neq x \in R$ define

$$N(x) = \max\left\{ r \mid \exists \text{ ideals } \{I_i\}_{i=1}^{r} \text{ of } R \text{ st } Rx = I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_r = R \right\}$$

So $N(x) \geq 1$

Define $N(0) = 0$

By constr N is positive norm on R

Show N is Dedekind-Hasse

Given nonzero $a, b \in R$ st $a \notin Rb$

Ideal $Ra + Rb$ is principal

Write $Ra + Rb = Rc$ $\quad c \in R$

Now

$$\underset{\substack{\uparrow \\ \text{since } c \neq 0}}{0 <} N(c) \underset{\substack{\uparrow \\ \text{since } Rb \subsetneq Rc}}{<} N(b)$$

$\square$

(ii) $\to$ (i) Given an ideal $I \subseteq R$

Display $b \in R$ st $I = Rb$

WLOG $I \neq 0$, else take $b = 0$

Define

$$m = \min \{ N(x) \mid x \in I, \quad x \neq 0 \}$$

$m > 0$ since N positive

Pick

$0 \neq b \in I$, $N(b) = m$

Show $Rb = I$

Suppose $Rb \subsetneq I$

$\exists$ $a \in I$, $a \notin Rb$

Since N is DH, $\exists$ $c \in Ra + Rb$ st

$$0 < N(c) < N(b)$$

By constr

$c \in I$, $c \neq 0$, $N(c) < m$ contr.

So $Rb = I$

$\square$

Recall the ring

$$R = \mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$$

We saw earlier that R is not a Eucl domain

Turns out R is a PID.

To show this, it suffices to display a Dedekind-Hasse norm N. The norm N is the usual one

$$N(x) = a^2+b^2 \qquad x = a+bi \qquad a, b \in \mathbb{R} \quad i^2=-1$$

The proof is a bit technical; see p 282 in text.