Lecture 42 Wednesday May 4

Examples involving Jordan canon form etc

Take our field $F = \mathbb{C}$ (complex numbers)

Ex 1 let $A = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}$

Find the JCF J for A.

Find an invertible matrix P such that

$$P^{-1} A P = J$$

Sol First find eigvalues of A

$$|xI - A| = \begin{vmatrix} x-6 & 7 \\ -1 & x+2 \end{vmatrix}$$

$$= (x-6)(x+2) + 7$$

$$= x^2 - 4x - 5$$

$$= (x-5)(x+1) \quad \text{char poly}$$

A has eigenvals

$$5, -1$$

A has min poly

$$(x-5)(x+1)$$

A is diagonalizable

$$J = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$$

Find eigenvectors for A to get P

eigval λ	5	-1
$A - \lambda I$	$\begin{pmatrix} 1 & -7 \\ 1 & -7 \end{pmatrix}$	$\begin{pmatrix} 7 & -7 \\ 1 & -1 \end{pmatrix}$
Solve $(A - \lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\overset{rr}{\Rightarrow} \begin{pmatrix} 1 & -7 \\ 0 & 0 \end{pmatrix}$ $x - 7y = 0$ $y = t$ free $x = 7t$ $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 7 \\ 1 \end{bmatrix}$	$\overset{rr}{\Rightarrow} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ $x - y = 0$ $y = t$ free $x = t$ $\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
eigvector for λ	$\begin{bmatrix} 7 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Define $P = \begin{bmatrix} 7 & 1 \\ 1 & 1 \end{bmatrix}$

By constr $AP = PJ$

So $P^{-1}AP = J$

$\square$

Ex 2 For above $A = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}$

Find RCF C for A

Find invertible matrix Q such that

$$Q^{-1} A Q = C$$

Sol A has single invariant factor

$$(x-5)(x+1) = x^2 - 4x - 5$$

$$A^2 = 5I + 4A$$

$$C = \begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix}$$

Find Q: Pick any vector $0 \neq u \in \mathbb{C}^2$ that is not an eigenvector for A, ie u not a scalar multiple of $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Say $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Obs $Au = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ -3 \end{pmatrix}$

Define

$$Q = \begin{pmatrix} 1 & -8 \\ 2 & -3 \end{pmatrix}$$

By constr

$$AQ = QC$$

So

$$Q^{-1}AQ = C$$

$\square$

Ex 3 For above $A = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix}$

apply row/col ops to put $xI - A$ in Smith normal form

Sol:

$$\begin{pmatrix} x-6 & 7 \\ -1 & x+2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & x+2 \\ x-6 & 7 \end{pmatrix} \qquad r_1 \leftrightarrow r_2$$

$$\begin{pmatrix} 1 & -x-2 \\ x-6 & 7 \end{pmatrix} \qquad r_1' = -r_1$$

$$\begin{pmatrix} 1 & -x-2 \\ 0 & x^2-4x-5 \end{pmatrix} \qquad r_2' = r_2 - (x-6)r_1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & x^2-4x-5 \end{pmatrix} \qquad c_2' = c_2 + (x+2)c_1$$

Ex 4 For

$$A = \begin{pmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

Find JCF J for A

Find invertible matrix P such that

$$P^{-1} A P = J$$

Sol Find eigvals for A

$$|xI - A| = \begin{vmatrix} x+2 & 9 & 0 \\ -1 & x-4 & 0 \\ -1 & -3 & x-1 \end{vmatrix}$$

$= (x-1)^3$ (skip details)
$=$ charpoly

eigvals are

1, 1, 1

A has min poly

$(x-1)^2$ or $(x-1)^3$

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Find a basis for the eigenspace for A with eigenvalue $\lambda = 1$

Solve $$(A - \lambda I)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A - \lambda I = \begin{pmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix} \overset{rr}{\Rightarrow} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Backsolve

$z = t$ free
$y = s$ free
$x = -3s$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eigenspace has basis

$$\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (\star)$$

dim = 2

J has two Jordan blocks

$$J = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

The inv factors of A are

$$x-1, \quad \underset{\substack{\shortparallel \\ \text{min poly}}}{(x-1)^2}$$

Find P

Find lin indep vectors v_1, v_2 such that

$$Av_1 = \lambda v_1 \qquad\qquad \lambda = 1$$
$$Av_2 = \lambda v_2 + v_1$$

Obs $\quad v_1 = (A - \lambda I) v_2$

$$\in \text{Col Space } (A - \lambda I)$$

$$= \text{Span} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

Take

$$v_1 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \qquad (= \text{sum of vectors } \star)$$

Find v_2

Write

$$v_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Require

$$(A - \lambda I)\, v_2 = v_1$$

$$\begin{bmatrix} -3 & -9 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$a + 3b = 1$$

Take any sol, say

$$a = 1, \qquad b = 0, \qquad c = 0$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Let v_3 = any linear combin of vectors in ☆ that is not a scalar multiple of v_1

Take for instance

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\mathbb{C}^3$ has basis

$$v_1, \quad v_2, \quad v_3$$

Let the matrix P have cols v_1, v_2, v_3:

$$P = \begin{pmatrix} -3 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Then

$$AP = PJ$$

So

$$P^{-1}AP = J$$

$\square$

Ex 5 For above

$$A = \begin{pmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

Find RCF C for A

Find invertible matrix Q such that

$$Q^{-1} A Q = C$$

SOL A has invariant factors

$$x-1, \quad (x-1)^2 = x^2 - 2x + 1$$

$$C = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 2 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Find Q: Pick any $u_1 \in \mathbb{C}^3$ that is not a lin comb of vectors ☆

so u_1 is not an eigenvector for A

Take for instance

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Obs

$$(A - \lambda I)\, u_1 \neq 0$$

$$(A - \lambda I)^2\, u_1 = 0$$

Take

$$u_2 = A u_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

obs

$$\begin{aligned} A u_2 &= A^2 u_1 \\ &= 2A u_1 - u_1 \\ &= 2u_2 - u_1 \end{aligned}$$

Take

$$u_3 = v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\mathbb{C}^3$ has basis

$$u_1, \quad u_2, \quad u_3$$

Let the matrix φ have cols u_1, u_2, u_3:

$$\varphi = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

By constr $\quad A\varphi = \varphi C \quad$ so $\quad \varphi^{-1} A \varphi = C$

$\square$

Ex 6 For above

$$A = \begin{pmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

Use row/col ops to put $xI - A$ in Smith normal form.

Sol:

$$\begin{pmatrix} x+2 & 9 & 0 \\ -1 & x-4 & 0 \\ -1 & -3 & x-1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & x-4 & 0 \\ x+2 & 9 & 0 \\ -1 & -3 & x-1 \end{pmatrix} \qquad r_1 \leftrightarrow r_2$$

$$\begin{pmatrix} 1 & 4-x & 0 \\ x+2 & 9 & 0 \\ -1 & -3 & x-1 \end{pmatrix} \qquad r_1' = -r_1$$

$$\begin{pmatrix} 1 & 4-x & 0 \\ 0 & x^2-2x+1 & 0 \\ 0 & 1-x & x-1 \end{pmatrix} \qquad \begin{aligned} r_2' &= r_2 - (x+2)r_1 \\ r_3' &= r_3 + r_1 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & (x-1)^2 & 0 \\ 0 & 1-x & x-1 \end{pmatrix} \qquad c_2' = c_2 - (4-x)c_1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-x & x-1 \\ 0 & (x-1)^2 & 0 \end{pmatrix} \qquad r_2 \leftrightarrow r_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x-1 & 1-x \\ 0 & (x-1)^2 & 0 \end{pmatrix} \qquad r_2' = -r_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x-1 & 1-x \\ 0 & 0 & (x-1)^2 \end{pmatrix} \qquad r_3' = r_3 - (x-1)r_2$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & (x-1)^2 \end{pmatrix} \qquad c_3' = c_3 - c_2$$

$\square$