Lecture 41 Monday May 2

The Jordan Canonical Form

Let F denote a field

Let ~~0 ≠~~ V denote a fin dim'l vector space over F

Given lin trans $T: V \to V$

For an indeterminate x, recall polynomial ring $F[x] = R$

View V as an $F[x]$-module with

$x v = T v \qquad \forall v \in V$

$F[x]$-module V is torsion since $\dim V < \infty$

$\mathrm{Ann}(V) = R_{m(x)}$

$m(x)$ = min poly of T

$m(x) \mid c(x) \qquad c(x)$ = char poly of T

Consider the elementary divisor decomp

of the R-module V.

To keep things simple we always assume:

Each root of $c(x)$ is contained in F $\qquad *$

(ie each eigenvalue of T is contained in F)

By the elem. divisor decomp,

V is a direct sum of finitely many cyclic

R-submodules, each with annihilator generated by

a prime power in R

Given a summand

$$Rv \qquad 0 \neq v \in V$$

$\mathrm{Ann}(Rv)$ is generated by a prime power in R, denoted

$$p^r \qquad r \geq 1 \qquad p \text{ prime (and monic)}$$

Recall $p \mid c(x)$ so by $*$,

$$p = x - \lambda \qquad \lambda \in F$$

LEM 1 With above notation,

the vector space Rv has a basis

$$v, \quad (x-\lambda)v, \quad (x-\lambda)^2 v, \quad \dots, \quad (x-\lambda)^{r-1} v \qquad **$$

pf Show $**$ spans Rv:

Rv is spanned by

$$v, \quad xv, \quad x^2 v, \quad \dots$$

Change vars $x \to x - \lambda$

Rv is spanned by

$$v, \quad (x-\lambda)v, \quad (x-\lambda)^2 v, \quad \dots$$

By cnstr

$$(x-\lambda)^r v = 0$$

So $**$ spans Rv ✓

Check ** lin indep :

Given $\alpha_i \in F$ $(0 \le i \le r-1)$ st

$$0 = \sum_{i=0}^{r-1} \alpha_i (x-\lambda)^i v$$

show $\alpha_i = 0$ for $0 \le i \le r-1$

Define $f(x) = \sum_{i=0}^{r-1} \alpha_i (x-\lambda)$

$f(x) v = 0$

$f(x) Rv = R f(x) v$ $\qquad R = F[x]$

$= 0$

$f(x) \in \mathrm{Ann}(Rv)$

$\underset{\deg \le r-1}{\underset{\uparrow}{f(x)}}$ is divisible by $\underset{\deg r}{\underset{\uparrow}{(x-\lambda)^r}}$

So $f(x) = 0$ in R

ie $\alpha_i = 0$ $0 \le i \le r-1$

$\square$

It is convenient to put ** in reverse order.

LEM 2 With above notation, with respect to the basis

$$(x-\lambda)^{r-1}v, \quad \ldots, \quad (x-\lambda)^2 v, \quad (x-\lambda)v, \quad v$$

the matrix representing T is

$$T: \begin{pmatrix} \lambda & 1 & & & & \\ & \lambda & 1 & & 0 & \\ & & \lambda & & & \\ & 0 & & \ddots & & \\ & & & & \lambda & 1 \\ & & & & & \lambda \end{pmatrix} \quad r \times r \qquad ☆$$

pf Define

$$v_i = (x-\lambda)^{r-1-i} v \qquad 0 \le i \le r-1$$

Obs

$$(X-\lambda I)\, v_i = v_{i-1} \qquad (1 \le i \le r-1),$$

$$(X-\lambda I)\, v_0 = 0$$

So

$$X v_i = \lambda v_i + v_{i-1} \qquad (1 \le i \le r-1)$$

$$X v_0 = \lambda v_0$$

X acts as T so result follows. $\square$

DEF 3 Referring to LEM 2, the matrix ☆ is called the <u>$r \times r$ Jordan block with eigenvalue λ</u>

— o —

Return to orig vs V

Thm 4 The vs V has a basis wrt which the matrix representing T is

$$T: \begin{pmatrix} \boxed{J_1} & & & 0 \\ & \boxed{J_2} & & \\ & & \ddots & \\ 0 & & & \boxed{J_s} \end{pmatrix}$$

"Jordan Canonical Form"

where each of $J_1, J_2, \ldots, J_s$ is a Jordan block

pf By LEM 2, and since V is a direct sum of cyclic R-submodules $\square$

Note 5

Referring to Thm 4, we can recover the elementary divisors of T from its Jordan canonical form.

Given an $r \times r$ Jordan block J with eigenvalue λ

$$
\begin{aligned}
(x-\lambda)^r &= \text{min poly of } J \\
&= \text{char poly of } J \\
&= \text{unique elem divisor for } J
\end{aligned}
$$

The elem divisors of T are

$$f_1, f_2, \ldots, f_s$$

where $\quad f_i = \text{elem divisor for } J_i \qquad 1 \le i \le s$

We clarify the uniqueness of the Jordan canonical form.

Thm 6 Referring to Thm 4, suppose the vs V has a second basis wrt which the matrix rep T is

$$T : \begin{pmatrix} \boxed{J_1'} & & & 0 \\ & \boxed{J_2'} & & \\ & & \ddots & \\ 0 & & & \boxed{J_S'} \end{pmatrix}$$

where each of $J_1', J_2', \dots, J_S'$ is a Jordan block.

Then $S = s$ and

$$J_1', J_2', \dots, J_s'$$

is a perm of

$$J_1, J_2, \dots, J_s$$

pf By Note 5 and the uniqueness of the elem divisors for T. $\square$

Ex 7 Given distinct $a, b, c \in F$

Assume T has Jordan canonical form

$$\begin{pmatrix}
\boxed{a} & & & & & & \\
& \boxed{\begin{matrix} a & 1 \\ 0 & a \end{matrix}} & & & & \Large 0 & \\
& & \boxed{\begin{matrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{matrix}} & & & & \\
& & & \boxed{\begin{matrix} b & 1 \\ 0 & b \end{matrix}} & & & \\
& & & & \boxed{\begin{matrix} b & 1 \\ 0 & b \end{matrix}} & & \\
& \Large 0 & & & & \boxed{\begin{matrix} c & 1 \\ 0 & c \end{matrix}} & \\
& & & & & & \boxed{\begin{matrix} c & 1 & 0 \\ 0 & c & 1 \\ 0 & 0 & c \end{matrix}}
\end{pmatrix}$$

Then the elem divisors of T are:

$x-a$, $(x-a)^2$, $(x-a)^3$,

$(x-b)^2$, $(x-b)^2$,

$(x-c)^2$, $(x-c)^3$

The inv factors of T are

$$x-a,$$

$$(x-a)^2(x-b)^2(x-c)^2,$$

$$(x-a)^3(x-b)^2(x-c)^3.$$

The minimal poly of T is

$$(x-a)^3(x-b)^2(x-c)^3.$$

The char poly of T is

$$(x-a)^6(x-b)^4(x-c)^5$$

DEF 8 Call T <u>diagonalizable</u> whenever V has a basis consisting of eigenvectors for T.

Thm 9 With the above notation TFAE

(i) T is diagonalizable.

(ii) the Jordan canonical form of T is diagonal

(iii) Each Jordan block for T is 1×1

(iv) the min poly of T has no repeated roots

pf: routine $\square$