LECTURE 39 Wednesday April 27

The Rational Canonical Form

Let F denote a field

For an indeterminate x, recall the polynomial ring $F[x]$.

We saw the ring $F[x]$ is a PID

Let V denote an $F[x]$-module

Then V is an F-module (ie a vector space over F)

and the map

$$\begin{aligned} T : \quad & V \longrightarrow V \\ & v \longrightarrow xv \end{aligned}$$

is an F-linear transformation

Assume the $F[x]$-module V is f.g.

Recall that V is the direct sum of finitely many cyclic $F[x]$-submodules.

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Assume the $F[x]$-module V is cyclic

and non 0.

So $\exists$ $0 \neq v \in V$ st

$$V = F[x]v$$

So the vector space V is spanned by

$$v, \quad Tv, \quad T^2v, \quad \ldots \qquad *$$

__Case:__ the $F[x]$-module V is free.

the vectors $*$ form a basis for vs V

The vs V has dim ∞

__Case:__ the $F[x]$-module V is torsion.

Write

$$Ann(V) = F[x]\, m(x) \qquad 0 \neq m(x) \in F[x]$$

polynomial $m(x)$ is unique up to mult by non 0 scalar in F

WLOG $m(x)$ is monic

__Obs__ $m(x)$ is the monic poly of least degree such that

$$m(T) = 0$$

Call $m(x)$ the __minimal__ __polynomial__ for T

Write

$$m(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{k-1} x^{k-1} + x^k$$

$k \geq 1$, $\quad b_0, b_1, \ldots, b_{k-1} \in F$

Then the vectors

$$v, Tv, T^2 v, \ldots, T^{k-1} v \qquad **$$

form a basis for the vs V and

$$T^k v = -b_0 v - b_1 Tv - b_2 T^2 v - \cdots - b_{k-1} T^{k-1} v$$

So vs V has dim k.

Let A = matrix in $Mat_k(F)$ that represents T wrt basis $**$.

Then

$$A = \left(\begin{array}{ccccc|c} 0 & & & & & -b_0 \\ 1 & 0 & & \bigcirc & & -b_1 \\ & 1 & & & & -b_2 \\ & & \ddots & \ddots & & \vdots \\ & \bigcirc & & 1 & 0 & \\ & & & & 1 & -b_{k-1} \end{array}\right)$$

Call A the <u>companion</u> <u>matrix</u> for the poly $m(x)$

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LEM 1 Given an $F[x]$-module V

TFAE

(i) $F[x]$-module V is fg and Torsion

(ii) vector space V is $\dim(V) < \infty$

pf (i) → (ii) V is dir sum of finitely many cyclic $F[x]$-submodules, each of which is torsion and hence fin dim.

(ii) → (i) Write $n = \dim(V)$

Pick a basis $\{v_i\}_{i=1}^{n}$ for V

$$V = \sum_{i=1}^{n} F v_i$$

$$= \sum_{i=1}^{n} F[x] v_i$$

So $F[x]$-module V is fg

Now $F[x]$-module V is dir sum of cyclic $F[x]$-submodules. None are free so all are torsion

So $F[x]$-module V has rank 0 ie V is torsion ☐

Until further notice the $F[x]$-module V satisfies

LEM 1 (i), (ii)

Consider the invariant factor decomp of the $F[x]$-module V.

the inv factors are (monic) polynomials in $F[x]$:

$$m_1(x), \quad m_2(x), \ldots, m_r(x)$$

each with degree ≥ 1 and

$$m_1(x) \mid m_2(x) \mid \cdots \mid m_r(x)$$

Recall

$$\mathrm{Ann}(V) = F[x]\, m_r(x)$$

$m_r(x)$ is monic poly of least degree st "min poly of T"

$$m_r(T) = 0$$

thm 2 With above notation,

$\exists$ basis for vs V with respect to which the matrix representing T is

$$T : \begin{pmatrix} \boxed{C_1} & & & 0 \\ & \boxed{C_2} & & \\ & & \ddots & \\ 0 & & & \boxed{C_r} \end{pmatrix}$$

"Rational Canonical Form"

where

C_i is the companion matrix of $m_i(x)$ $1 \le i \le r$

pf $F[x]$-module V is ds of cyclic $F[x]$-submodules

$$V = F[x]v_1 + F[x]v_2 + \cdots + F[x]v_r \quad \text{ds}$$

with

$$\mathrm{Ann}\left(F[x]v_i\right) = F[x]\, m_i(x) \quad 1 \le i \le r$$

For $1 \le i \le r$ vs $F[x]v_i$ has basis wrt which matrix rep T is C_i. Result follows. □

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DEF 3 For $\lambda \in F$ define

$$V_\lambda = \{ v \in V \mid Tv = \lambda v \}$$

Call λ an eigenvalue of T iff $V_\lambda \neq 0$

In this case call V_λ the λ-eigenspace for T

LEM 4 Pick a basis for V: $\quad *$

$v_1, v_2, \dots v_n$

Let B denote the matrix rep T wrt $*$.

Then $\det(B)$ is indep of $*$

pf Pick a second basis for V: $\quad **$

$w_1, w_2, \dots, w_n$

Let $S \in \mathrm{Mat}_n(F)$ denote the transition matrix from $*$ to $**$. The matrix rep T wrt $**$ is

$$S^{-1} B S$$

Now
$$\det(S^{-1}BS) = \det(S)\det(B)\underset{\det(S)^{-1}}{\underset{\shortparallel}{\det(S^{-1})}}$$
$$= \det(B) \checkmark$$

$\square$

DEF 5 By the <u>determinant of T,</u> we mean $\det(B)$ where the matrix B represents T wrt some basis for V.

Prop 6 For $\lambda \in F$ TFAE

(i) λ is an eigenvalue of T

(ii) $\lambda I - T$ is not invertible

(iii) $\det(\lambda I - T) = 0$

pf elem. lin. alg. $\square$

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Def 7 The *characteristic polynomial* of T is the following poly in $F[x]$:

$$\underset{\overset{\|}{C(x)}}{\det(xI - T)} \qquad *$$

obs $*$ is monic with degree $= \dim(V)$

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We now describe how the char poly $C(x)$ is related to the $m_T(x)$

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Prop 8 With above notation

$$c(x) = m_1(x)\, m_2(x) \cdots m_r(x)$$

pf Consider the basis for V as in Thm 2.

Rel this basis

$$T : \begin{pmatrix} \boxed{C_1} & & & O \\ & \boxed{C_2} & & \\ & O & \ddots & \\ & & & \boxed{C_r} \end{pmatrix}$$

so

$$xI - T : \begin{pmatrix} \boxed{xI-C_1} & & & \\ & \boxed{xI-C_2} & & \\ & & \ddots & \\ & & & \boxed{xI-C_r} \end{pmatrix}$$

$$\det(xI - T) = \prod_{i=1}^{r} \underbrace{\det(xI - C_i)}_{\substack{\| \text{ ex} \\ m_i(x)}}$$

Cor 9. the $\overset{m(x)}{\overset{\shortparallel}{\text{min poly}}}$ of T divides the $\underset{c(x)}{\underset{\shortparallel}{\text{char poly}}}$ of T.

pf. We saw $m(x) = m_r(x)$

By Prop 8, $m_r(x) \mid c(x)$ □

Cor 10 (Cayley-Hamilton thm)

$$c(T) = 0$$

where $c(x)$ is the char poly of T.

pf $$c(T) = m_1(T)\, m_2(T) \cdots \underbrace{m_r(T)}_{\overset{\shortparallel}{0}}$$

$$= 0$$

Cor 11 the $\overset{C(x)}{\text{char poly}}$ of T divides some power of the $\underset{m(x)}{\text{min poly}}$ of T.

pf By Prop 8

$$C(x) = m_1(x)\, m_2(x) \cdots \overset{m(x)}{m_r(x)}$$

Fn $1 \le i \le r$

$$m_i(x) \mid m(x)$$

Write

$$m(x) = m_i(x)\, M_i(x) \qquad M_i(x) \in F[x]$$

obs

$$m(x)^r = m_1(x) M_1(x)\; m_2(x) M_2(x) \cdots m_r(x) M_r(x)$$

$$= \underbrace{m_1(x)\, m_2(x) \cdots m_r(x)}_{C(x)}\; M_1(x) \cdots M_r(x)$$

So $C(x) \mid m(x)^r$