Lecture 38 Monday April 25

Assume R is a PID

Our goal: For fg R-modules, show the uniqueness of the invariant factors and elementary divisors

Let U denote a fg R-module of rank t

We saw $\exists$ R-submodule F of U st both

(i) the sum $U = Tor(U) + F$ is direct

(ii) $\exists$ R-module iso $F \simeq R^t$

The inv factors and elem divisors of U both describe how to decompose $Tor(U)$ as a direct sum of cyclic R-modules. To show the inv factors and elem divisors are unique, WLOG U is torsion.

Assum U is torsion and non 0

Write $\operatorname{Ann}(U) = Rp$ $\quad 0 \neq p \in R$

Factor p into irreducibles:

up to associate

$$p = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} \qquad r, e_1, e_2, \ldots, e_r > 0$$

p_i, p_j not associates if $i \neq j$ $\quad (1 \leq i, j \leq r)$

For $1 \leq i \leq r$ define

$$U(i) = \{ v \in U \mid \exists\ n \geq 1 \text{ st } p_i^n v = 0 \}$$

One checks $U(i)$ is an R-submodule of U

Call $U(i)$ the <u>p_i - primary component</u> of U

One checks

$$\operatorname{Ann}(U(i)) = R p_i^{e_i}$$

By thm 27 (or directly from CRT REM thm),

$$U = U(1) + U(2) + \cdots + U(r) \qquad \text{dir sum}$$

LEM 28 Given a prime $p \in R$

and obs the R-module R/Rp is a field
$$R/Rp = \mathbb{F}$$

Given a fg R-module V.

(i) Assume $V = R^t$ is free. Then

$\exists$ R-module iso
$$V/pV \simeq \mathbb{F}^t$$

(ii) Assume $V = R/Ra$ $0 \neq a \in R$.

then $\exists$ R-module iso
$$V/pV \simeq \begin{cases} \mathbb{F} & \text{if } p \mid a \\ 0 & \text{if } p \nmid a \end{cases}$$

(iii) Assume
$$V = R/Ra_1 \times R/Ra_2 \times \cdots \times R/Ra_k$$

with $p \mid a_i$ for $1 \le i \le k$.

then $\exists$ R-module iso
$$V/pV \simeq \mathbb{F}^k$$

pf (i) The map

$$\varphi: V \longrightarrow \mathbb{F}^t$$

$$(x_1, x_2, \ldots, x_t) \longrightarrow (x_1 + Rp, x_2 + Rp, \ldots, x_t + Rp)$$

is surj R-mod hom.

Obs

$$\ker(\varphi) = pV$$

So φ induces R-module iso

$$V/pV \cong \mathbb{F}^t$$

(ii) Write $W = V/pV$

Suppose $p \nmid a$

By constr $pW = 0$ and $aW = 0$

We have $GCD(p, a) = 1$, so $1W = 0$

so $V/pV = 0$

Suppose $p \mid a$

so $Ra \leq Rp$

$\exists$ surj R-module hom

$$\varphi : \underbrace{R/Ra}_{=V} \longrightarrow \underbrace{R/Rp}_{=\mathbb{F}}$$

$$r + Ra \longrightarrow r + Rp$$

We have $\ker(\varphi) = pV$

So φ induces R-mod iso

$$V/pV \cong \mathbb{F}$$

(iii) Use (ii) $\square$

LEM 29 Given a prime $p \in R$

Given pos integers

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_s \qquad 0 \le s, t < \infty$$

$$\beta_1 \le \beta_2 \le \cdots \le \beta_t$$

TFAE

(i) these R-modules are iso:

$$V = R/Rp^{\alpha_1} \times R/Rp^{\alpha_2} \times \cdots \times R/Rp^{\alpha_s},$$

$$W = R/Rp^{\beta_1} \times R/Rp^{\beta_2} \times \cdots \times R/Rp^{\beta_t}$$

(ii) $s = t$ and

$$\alpha_i = \beta_i \qquad 1 \le i \le s$$

pf (i) → (ii)

$$\mathrm{Ann}(V) = Rp^{\alpha_s}$$
$$\parallel$$
$$\mathrm{Ann}(W) = Rp^{\beta_t}$$

$*$

So $\alpha_s = \beta_t$

We proceed by induction on $*$

Case $* = 1$ Here

$$\alpha_i = 1 \qquad 1 \le i \le s$$
$$\beta_i = 1 \qquad 1 \le i \le t$$

So $V = \mathbb{F}^s$, $W = \mathbb{F}^t$

$\mathbb{F} = R/Rp$

Viewing V, W as free $\mathbb{F}$-modules (ie vs over $\mathbb{F}$)

we see $s = t$ ✓

Case $* \geq 2$ We have R mod iso

$pV \cong pW$

Obs $pV \cong$

$$\underbrace{R/Rp^{\alpha_1 - 1}}_{\substack{\parallel \\ 0 \text{ if } \alpha_1 = 1}} \times R/Rp^{\alpha_2 - 1} \times \cdots \times R/Rp^{\alpha_s - 1}$$

Also $pW \cong$

$$R/Rp^{\beta_1 - 1} \times R/Rp^{\beta_2 - 1} \times \cdots \times R/Rp^{\beta_t - 1}$$

Consider the sequences

$$\alpha_1 - 1 \le \alpha_2 - 1 \le \cdots \le \alpha_s - 1 \qquad ☆$$

$$\beta_1 - 1 \le \beta_2 - 1 \le \cdots \le \beta_t - 1 \qquad ☆☆$$

By ind,

sequence of non 0 terms in ☆

= sequence of non 0 terms in ☆☆.

Also $s = t$ since by LEM 28,

$$\begin{array}{ccc} V/pV & \cong & W/pW \\ \cong & & \cong \\ \mathbb{F}^s & & \mathbb{F}^t \end{array}$$

Now these sequence coincide:

$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_s$

$\beta_1 \le \beta_2 \le \cdots \le \beta_t$

(ii) → (i) clear. □

Thm 30 Given fg R-modules U, V

TFAE

(i) $\exists$ R-module iso $U \cong V$

(ii) U, V have the same free-rank and same list of elem divisors (up to perm + assoc)

pf (i) $\to$ (ii) We saw free-rank = rank, so these are same.

We have $Tor(U)$ iso $Tor(V)$

WLOG $U = Tor(U)$, $V = Tor(V)$

Recall

U = ds of its primary components

V = ...

So for each prime $p \in R$

p-primary comp of U $\cong$ p-primary comp of V

WLOG $\exists$ prime $p \in R$ s/t

U = its p-primary comp

V = ...

Now U, V have same elem divisors by LEM 29.

(ii) $\to$ (i) ✓ $\square$

Thm 31 Given fg R-modules U, V

TFAE

(i) $\exists$ R-module iso $U \cong V$

(ii) U, V have same free rank and same list of invariant factors (up to assoc)

pf The inv factors and elem divisors determine each other. Result follows by Thm 30. $\square$