Lecture 36 Wednesday April 20

Recall R is a commutative ring with $1 \neq 0$

LEMMAS 23, 24 are special cases of the following result.

Theorem 25 Assume R is a PID

Consider a free R-module R^n

Given an R-submodule W of R^n

Write $m = \text{rank}(W)$

Then $\exists$ linearly independent generators $v_1, v_2, \ldots, v_n$ for R^n and nonzero $d_1, d_2, \ldots, d_m$ in R such that both

(i) $d_1 \mid d_2 \mid \cdots \mid d_m$

(ii) $W = R d_1 v_1 + R d_2 v_2 + \cdots + R d_m v_m$

pf Use induction on n

<u>Case</u> $n = 0$ ✓

<u>Case</u> $n \geq 1$:

Assume $W \neq 0$ else $m=0$ and result is trivial

So $m \geq 1$.

Recall

$$\mathrm{Hom}_R(R^n, R) = \{ \varphi \mid \varphi : R^n \to R \text{ is an } R\text{-module hom} \}$$

$\mathrm{Hom}_R(R^n, R)$ contains

$$\begin{aligned} 0 : \; & R^n \to R \\ & x \to 0 \end{aligned}$$

$\forall \varphi \in \mathrm{Hom}_R(R^n, R)$

$\varphi(W)$ is an R-submodule of R,

in other words an ideal of the ring R

Define the set

$$\Sigma = \{ \varphi(W) \mid \varphi \in \mathrm{Hom}_R(R^n, R) \}$$

Σ contains the 0 ideal of R

<u>claim I</u> : Σ contains an element J that is not properly contained in any element of Σ

"J is maximal"

<u>pf I</u> Suppose not. Σ is nonempty, so

pick $J_1 \in \Sigma$

$\exists$ $J_2 \in \Sigma$ st $J_1 \subsetneq J_2$

$\exists$ $J_3 \in \Sigma$ st $J_2 \subsetneq J_3$

...

the sequence

$$J_1 \subsetneq J_2 \subsetneq \cdots$$

contradicts the fact that the R-module R is Noetherian ✓

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the ideal J is principal; write

$$J = R d_1 \qquad d_1 \in R$$

Pick

$$\nu \in \mathrm{Hom}_R(R^n, R)$$

st

$$J = \nu(W)$$

Pick $y \in W$ st

$$d_1 = \nu(y)$$

<u>claim II</u> : $d_1 \neq 0$

<u>pf II</u> Pick $0 \neq w \in W$

Write
$$w = (\alpha_1, \alpha_2, \dots, \alpha_n) \qquad \alpha_i \in R$$

$\exists\, i \ (1 \le i \le n)$ st $\alpha_i \neq 0$

The map
$$\varphi : \begin{array}{ccc} R^n & \longrightarrow & R \\ (a_1, a_2, \dots, a_n) & \longrightarrow & a_i \end{array}$$

is an R-module hom, so
$$\varphi \in Hom_R(R^n, R)$$

So $\varphi(W) \in \Sigma$

By constr $\varphi(W) \neq 0$

Now $d_1 \neq 0$, else
$$Rd_1 \subsetneq \varphi(W) \qquad \text{a contradiction.}$$

claim III : For $\varphi \in \mathrm{Hom}_R(R^n, R)$,

$$d_1 \text{ divides } \varphi(y)$$

pf III Suppose not. Then

$$\varphi(y) \notin Rd_1 = J$$

So $\varphi(y) \neq 0$

Define $d = GCD(d_1, \varphi(y))$

So $Rd = \underset{\overset{\parallel}{J}}{Rd_1} + R\varphi(y)$

Obs

$$J \subsetneq Rd$$

$\exists$ $r, s \in R$ such that

$$d = \underset{\overset{\parallel}{\nu(y)}}{rd_1} + s\varphi(y)$$

Define $\psi = r\nu + s\varphi \in \mathrm{Hom}_R(R^n, R)$

Obs $d = \psi(y)$

Now $J \subsetneq Rd = R\psi(y) = \psi(Ry) \leq \underset{\overset{\cap}{\Sigma}}{\psi(W)}$

This contradicts the maximality of J from claim I.

<u>Claim IV</u> : $\exists \; v_1 \in R^n$ such that $y = d_1 v_1$

<u>pf IV</u> For $1 \le i \le n$ the map

$$R^n \longrightarrow R \qquad *$$
$$(a_1, a_2, \ldots, a_n) \longrightarrow a_i$$

is an R-module hom

By Claim III, d_1 divides the image of y under $*$

So d_1 divides each coordinate of y.

Write
$$y = (d_1 b_1, d_1 b_2, \ldots, d_1 b_n) \qquad b_i \in R$$

Define
$$v_1 = (b_1, b_2, \ldots, b_n) \in R^n$$

By constr
$$y = d_1 v_1$$

✓

<u>claim V</u> : $\nu(v_1) = 1$

<u>pf V</u> We have

$$d_1 = \nu(y) = \nu(d_1 v_1) = d_1 \nu(v_1)$$

so $\underbrace{d_1}_{\neq 0} (1 - \nu(v_1)) = 0$

R is an integral domain so

$$1 - \nu(v_1) = 0$$

✓

<u>claim VI</u> : The sum $R^n = Rv_1 + \ker(\nu)$ is direct.

<u>pf VI</u> the R-module hom $\nu : R^n \to R$ is surjective by claim V. Claim follows via LEM 21

✓

<u>claim VII</u>: The sum

$$W = R d_1 v_1 + W \cap \ker(\nu)$$

is direct

<u>pf VII</u> the map

$$\nu|_W : \begin{array}{ccc} W & \to & R \\ w & \to & \nu(w) \end{array}$$

is an R-module hom.

Its image is

$$\nu(W) = J = R d_1$$

So

d_1 divides $\nu(w)$ $\forall\, w \in W$

For $w \in W$ $\exists$ element in R, denoted $\phi(w)$, st

$$\nu(w) = \phi(w)\, d_1$$

the map

$$\phi : \begin{array}{ccc} W & \longrightarrow & R \\ w & \to & \phi(w) \end{array}$$

is an R-module hom

Recall

$$\begin{array}{c} \nu(y) = d_1 \\ \| \\ \phi(y)\, d_1 \end{array}$$

So
$$\phi(y) = 1$$

so ϕ is surjective

Now by LEM 21 the sum

$$W = Ry + \ker(\phi)$$

is direct.

By construction
$$y = d_i v_i$$
$$\ker(\phi) = W \cap \ker(\nu)$$

the claim follows

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If $n = 1$ we are done.
For the rest of the pf assume $n \geq 2$

By Claim VI,

$$R^n = Rv_1 + \ker(\nu) \qquad \text{(dir sum)}$$

rank: n $\quad 1 \quad$ $n-1$

By Prop 22 $\exists$ R-module iso

$$\ker(\nu) \simeq R^{n-1}$$

By Claim VII,

$$W = R d_1 v_1 + W \cap \ker(\nu) \qquad \text{(dir sum)}$$

rank: m $\quad 1 \quad$ $m-1$

Obs

$W \cap \ker(\nu)$ is an R-submodule of $\ker(\nu)$

By induction, $\exists$ lin indep gens $v_2, v_3, \ldots, v_n$ for $\ker(\nu)$

and non0 $d_2, d_3, \ldots, d_m$ in R st both

(i) $d_2 | d_3 | \cdots | d_m$

(ii) $W \cap \ker(\nu) = R d_2 v_2 + R d_3 v_3 + \cdots + R d_m v_m$

By Constr

$v_1, v_2, \dots, v_n$ are lin indep gens for R^n

Also

$W = R d_1 v_1 + R d_2 v_2 + \dots + R d_m v_m$

<u>Claim VIII</u> $d_1 \mid d_2$

<u>pf VIII</u> $\exists$ R-module hom

$\varphi : R^n \to R$

that sends

$v_1 \to 1, \quad v_2 \to 1, \quad v_i \to 0 \quad (3 \le i \le n)$

Consider $\varphi(W) \in \Sigma$

$d_1 v_1 \in W, \quad d_2 v_2 \in W$

$\varphi(d_1 v_1) = d_1 \quad \varphi(d_2 v_2) = d_2$

$d_1, d_2 \in \varphi(W)$

$R d_1 + R d_2 \subseteq \varphi(W)$

$\overset{\shortparallel}{J}$

$d_2 \in R d_1$ else $J \subsetneq \varphi(W)$ cont.

So $d_1 \mid d_2$

$\square$