Lecture 34 Friday April 15

Recall R is a commutative ring with $1 \neq 0$

For R-modules, we continue to discuss rank and linear independence.

Assume R is an integral domain.

Given an R-module V and elements $\{v_i\}_{i=1}^n$ in V. Consider the R-submodule

$$W = \sum_{i=1}^n R v_i$$

Consider the map

$$\varphi : R^n \longrightarrow W$$

$$(c_1, c_2, \ldots, c_n) \longrightarrow c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

obs φ is a surjective R-module hom

Moreover TFAE

(i) $\{v_i\}_{i=1}^n$ are lin indep.

(ii) φ is R-module iso.

LEM 14 Assume R is an integral domain.

Given an R-module V and R-submodules $\mathcal{U}, \mathcal{U}'$ of V such that $\mathcal{U} \cap \mathcal{U}' = 0$

Given lin indep elements of $\mathcal{U}$:

$u_1, u_2, \ldots, u_n$ $\quad *$

Given lin indep elements of $\mathcal{U}'$:

$u_1', u_2', \ldots, u_m'$ $\quad **$

Then

$u_1, u_2, \ldots, u_n, u_1', u_2', \ldots, u_m'$

are lin indep in V.

pf Given $\{c_i\}_{i=1}^n$, $\{c_j'\}_{j=1}^m$ in R

st

$$\overbrace{\underbrace{\sum_{i=1}^n c_i u_i}_{\substack{\cap \\ \mathcal{U}}}}^{u} + \underbrace{\sum_{j=1}^m c_j' u_j'}_{\substack{\cap \\ \mathcal{U}'}} = 0$$

$u \in \mathcal{U} \cap \mathcal{U}' = 0$

$\{c_i\}_{i=1}^n$ all 0 since $*$ lin indep

$\{c_j'\}_{j=1}^m$ all 0 since $**$ lin indep $\square$

LEM 15 Assume R is an integral domain

Given an R-module V and linearly independent elements

$$u_1, u_2, \dots, u_n \in V$$

For $A \in \mathrm{Mat}_n(R)$ consider the elements

$$v_j = \sum_{i=1}^{n} A_{ij} u_i \qquad 1 \le j \le n \qquad *$$

then $*$ are lin indep iff $\det(A) \neq 0$

pf For

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in R^n$$

we have

$$\sum_{j=1}^{n} c_j v_j = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{n} A_{ij} u_i \right)$$

$$= \sum_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{n} A_{ij} c_j \right)}_{\substack{\parallel \\ \text{coord } i \text{ of } Ac}} u_i$$

so

$$\sum_{j=1}^{n} c_j v_j = 0 \quad \text{iff} \quad Ac = 0$$

Recall $Ac = 0$ has a nonzero sol for c iff $\det(A) \neq 0$

Result follows.

$\square$

COR 16 Assume R is an integral domain

Given an R-module V and lin indep elements

$$u_1, u_2, \ldots, u_n \in V$$

Given $0 \neq a_i \in R$ $\quad 1 \le i \le n$

then the elements

$$a_1 u_1, a_2 u_2, \ldots, a_n u_n$$

are lin indep.

pf The matrix

$$A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$$

has det

$$a_1 a_2 \cdots a_n \qquad *$$

$* \neq 0$ since R is integral domain.

Result follows by LEM 15. $\square$

Assume R is an integral domain.

Given an R-module V

Given lin indep elements

$$v_1, v_2, \ldots, v_n \in V \qquad n = \operatorname{rank}(V)$$

So the R-submodule

$$W = \sum_{i=1}^{n} R v_i$$

is iso R^n

LEM 17 With above notation, the quotient R-module V/W is torsion.

pf $\forall v \in V$ the elements

$v, \; v_1, v_2, \ldots, v_n$ are lin dep

So $\exists \; a, \; a_1, a_2, \ldots, a_n \in R$ not all 0

st $av + \sum_{i=1}^{n} a_i v_i = 0$

$a \neq 0$ since $v_1, v_2, \ldots, v_n$ are lin indep. Also

$$av \in \sum_{i=1}^{n} R v_i = W$$

so in V/W $a(v+W) = av + W = W$

So $v + W$ is torsion. $\square$

We now reverse the logical direction

LEM 18 Assume R is an integral domain.

Given an R-module V and $n \geq 0$

Assume $\exists$ R-submodule W of V that is

iso to R^n and V/W is torsion. Then

$$\underset{\substack{\| \\ N}}{\operatorname{rank}(V)} = n$$

pf obs

$$n = \operatorname{rank}(W) \leq \operatorname{rank}(V) = N$$

show $n \geq N$:

$\exists$ lin indep

$$v_1, v_2, \dots, v_N \in V$$

For $1 \leq i \leq N$ $\exists$ $0 \neq a_i \in R$ st

$a_i v_i \in W$ since V/W torsion

By Cor 16,

$a_1 v_1, a_2 v_2, \dots, a_N v_N$ lin indep

Now $n \geq N$ since W has rank n. $\square$

Prop 19 Assume R is an integral domain.

Given R-modules U, V.

Then for the direct product R-module $U \times V$,

$$\underbrace{\operatorname{rank}(U \times V)}_{N} = \underbrace{\operatorname{rank}(U)}_{m} + \underbrace{\operatorname{rank}(V)}_{n}$$

pf Write $W = U \times V$

view U, V as R-submodules of W, so

$$W = U + V \quad (\text{dir sum})$$

$\exists$ lin indep

$$u_1, u_2, \dots, u_m \in U$$

$\exists$ lin indep

$$v_1, v_2, \dots, v_n \in V$$

<u>Show $N \geq m + n$</u> : the sum $U + V$ is direct, so

by LEM 14,

$u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$ lin indep in W ✓

<u>Show $N \le m + n$</u> $\exists$ lin indep elements

$$w_1, w_2, \ldots, w_N \in W$$

For $1 \le i \le N$ write

$$w_i = \underset{\underset{U}{\cap}}{w_i^+} + \underset{\underset{V}{\cap}}{w_i^-}$$

obs

$w_i^+, \quad u_1, u_2, \ldots, u_m$ lin dep

So $\exists$ $0 \ne a_i \in R$ st

$$a_i w_i^+ \in \underbrace{Ru_1 + Ru_2 + \cdots + Ru_m}_{\overline{U}}$$

obs

$w_i^-, \quad v_1, v_2, \ldots, v_n$ lin dep

So $\exists$ $0 \ne b_i \in R$ st

$$b_i w_i^- \in \underbrace{Rv_1 + Rv_2 + \cdots + Rv_n}_{\overline{V}}$$

Obs

$$a_i b_i \neq 0 \quad \text{since} \quad R \text{ is int domain}$$

Obs

$$\begin{aligned} a_i b_i w_i &= a_i b_i (w_i^+ + w_i^-) \\ &= \underbrace{b_i (a_i w_i^+)}_{\in \bar{U}} + \underbrace{a_i (b_i w_i^-)}_{\in \bar{V}} \\ &\in \bar{U} + \bar{V} \end{aligned}$$

By Cor 16

$$a_i b_i w_i \qquad 1 \le i \le N \qquad (*)$$

are lin indep.

By above LEM 14 we have R module isomorphisms

$$\bar{U} \simeq R^m, \qquad \bar{V} \simeq R^n$$

So $\quad \bar{U} + \bar{V} \simeq R^{m+n}$

So

$$\operatorname{rank}(\bar{U} + \bar{V}) = m+n$$

But $*$ gives N lin indep elements in $\bar{U} + \bar{V}$.

So

$$N \le m+n$$

$\square$

Assume R is integral domain

Given fg R-module V

<u>Describe V</u>

Write $V = \sum_{i=1}^{n} R v_i$

Recall the map

$$\varphi : \begin{array}{ccc} R^n & \longrightarrow & V \\ (a_1, a_2, \ldots, a_n) & \longmapsto & a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \end{array}$$

is surj R-module homomorphism.

Let $W = \ker(\varphi)$

So W is R-submodule of R^n

φ induces R-module iso

$$\begin{array}{ccc} R^n / W & \longrightarrow & V \\ x + W & \longmapsto & \varphi(x) \end{array}$$

To describe the solutions for V, it suffices to describe the R-submodules W of R^n

Given R-submodule W of R^n

By LEM 13,

W is torsion-free

Write $m = \text{rank}(W)$

Obs $m \le \text{rank}(R^n) = n$

Natural question: does there exist an R-module iso

$$W \simeq R^m \quad ?$$

We will show: ans is "No" in general
ans is "yes" if R is a PID

The next example illustrates why the ans is NO
in general.