Lecture 33 Wednesday April 13

Recall R is a commutative ring with $1 \neq 0$

Assume R is an integral domain, and consider two **free** R-modules of finite rank:

$$U = R^m, \qquad V = R^n \qquad m \neq n$$

Conceivably the R-modules U, V are isomorphic.

LEM 10 The above R-modules U, V are not iso.

pf WLOG $m > n$. Suppose $\exists$ R-module iso

$$\varphi : U \to V$$

View elements in U, V as column vectors

Let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ e_i $\in U$ $\quad 1 \le i \le m$

$$\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_m) \in V$$

By LEM 9 $\exists$ $c_1, c_2, \ldots, c_m \in R$, not all 0, st

$$\sum_{i=1}^m c_i \varphi(e_i) = 0$$

$$\parallel$$

$$\varphi\left(\sum_{i=1}^m c_i e_i\right)$$

φ is bijection so

$$\sum_{i=1}^m c_i e_i = 0$$

Cont.

$\square$

Back to general R

Given R-module V

For $v \in V$ consider

$$\{ r \in R / \ rv = 0 \} \qquad *$$

$*$ is an ideal of R "order ideal of v"

Call v <u>torsion</u> if $* \neq 0$

<u>torsion-free</u> if $* = 0$

For $v = 0$; $* = R \neq 0$ so $v = 0$ is torsion

Define

$$Tor(V) = \{ v \in V / \ v \text{ is torsion} \}$$

LEM 11 Assume R is an integral domain.

Then for any R-module V,

$\mathrm{Tor}(V)$ is an R-submodule of V

pf For $u, v \in \mathrm{Tor}(V)$ show

$$u+v \overset{?}{\in} \mathrm{Tor}(V)$$

$\exists$ $0 \neq r \in R$ st $ru = 0$

$\exists$ $0 \neq s \in R$ st $sv = 0$

$$rs(u+v) = \underset{\underset{\underbrace{sru}_{=0}}{\parallel}}{rsu} + \underbrace{rsv}_{=0}$$

$$= 0$$

$rs \neq 0$ since R is int domain. ✓

For $v \in \mathrm{Tor}(V)$ and $r \in R$ show

$$rv \in \mathrm{Tor}(V)$$

$\exists$ $0 \neq s \in R$ st $sv = 0$

$$s(rv) = r\underbrace{sv}_{=0}$$

$$= 0$$ ✓

$\square$

Assume R is an integral domain.

Given an R-module V

Call V <u>torsion</u> if $Tor(V) = V$

<u>torsion-free</u> if $Tor(V) = 0$

LEM 12 Assume R is an integral domain. For an R-module V, the quotient R-module

$$V/\mathrm{Tor}(V)$$

is torsion-free.

pf Write $W = V/\mathrm{Tor}(V)$

For $w \in \mathrm{Tor}(W)$ show $w = 0$

$\exists\ 0 \neq r \in R$ st $rw = 0$

Write $w = v + \mathrm{Tor}(V)$ $v \in V$

show $v \in \mathrm{Tor}(V)$

Obs
$$\underset{\substack{\parallel \\ 0 + \mathrm{Tor}(W)}}{rw} = rv + \mathrm{Tor}(V)$$

So $rv \in \mathrm{Tor}(V)$

So $\exists\ 0 \neq s \in R$ st

$$\underset{\substack{\parallel \\ (sr)v}}{s(rv)} = 0$$

$sr \neq 0$ since R is int dom.

So $v \in \mathrm{Tor}(V)$

$\square$

LEM 13 Assume R is an integral domain.

Given a free R-module V of finite rank n:

$$V = R^n$$

Then V is torsion-free.

pf For $v \in Tor(V)$ show $v = 0$

$\exists\ 0 \neq r \in R$ st $rv = 0$

Write $v = (a_1, a_2, \dots, a_n)$ $a_i \in R$

For $1 \leq i \leq n$, $r a_i = 0$

Now $a_i = 0$ since R is int domain

Now $v = 0$ $\square$

— o —

Going to show:

Assuming R is a PID, and the R-module V is f.g.

V is free if and only if V is torsion free

For an R-module $V \neq 0$

define

$$Ann(V) = \{ r \in R \mid rv = 0 \ \forall v \in V \}$$

$\overset{\shortparallel}{A}$ "Annihilator of V"

A is an ideal of R

note $A \neq R$:

$1 \notin A$ since $1v = v$ $\forall v \in V$ and $V \neq 0$

Consider quot ring

$$\mathcal{R} = R/A$$

V becomes an $\mathcal{R}$-module with action

$$\mathcal{R} \times V \to V$$
$$r + A \quad v \to rv$$

One checks that for the $\mathcal{R}$-module V.

$$Ann(V) = 0$$

pf For $x \in Ann(V)$ show $x = 0$

Write $x = r + A$ $r \in R$

show $r \in A$

$\forall v \in V$, $0 = xv = (r + A)v = rv$

So $r \in A$

Assume R is an integral domain.

Given an R-module $V \neq 0$

Assume $\mathrm{Ann}(V) \neq 0$.

$\exists\ 0 \neq r \in \mathrm{Ann}(V)$

$rv = 0 \quad \forall\, v \in V$

V is torsion.

Conversely, assume V is torsion and f.g.

Let $S =$ finite gen set for V

So $V = \sum_{v \in S} Rv$

For $v \in S$, $\exists\ 0 \neq r_v \in R$ st $r_v v = 0$

Define $r = \prod_{v \in S} r_v$

$r \neq 0$ since R is int domain

Obs $rv = 0 \quad \forall v \in S$

So $rv = 0 \quad \forall v \in V$

so $r \in \mathrm{Ann}(V)$

Now $\mathrm{Ann}(V) \neq 0$

Given R-module V and R-submodule $U \subseteq V$

$\forall\, r \in R$,

$$rV = 0 \quad \rightarrow \quad rU = 0$$

So

$$\mathrm{Ann}(V) \subseteq \mathrm{Ann}(U) \qquad *$$

Now assume R is a PID

Write

$$\mathrm{Ann}(U) = Ra$$

$$\mathrm{Ann}(V) = Rb$$

By $*$

$$b \in Ra$$

So

$$a \text{ divides } b$$

Assume R is an integral domain.

Given R-module V

Given elements

$$v_1, v_2, \dots, v_n \qquad *$$

in V

Call $*$ <u>linearly independent</u> whenever for all

$$c_1, c_2, \dots, c_n \in R$$

$$\sum_{i=1}^{n} c_i v_i = 0$$

implies $\quad c_i = 0 \quad$ for $1 \le i \le n$

Define

$$\text{rank}(V) = \max\{n \mid \exists\ n \text{ lin indep elements in } V\}$$

Poss

$$\text{rank}(V) = \infty$$

For an R-submodule $U \subseteq V$ we have

$$\text{rank}(U) \le \text{rank}(V)$$

- The free R-module R^n has rank n in above sense.

- Each torsion R-module V has rank 0

Since $\forall$ $v \in V$

$\exists$ $0 \neq r \in R$ st $rv = 0$

So v not lin indep