Lecture 32 Monday April 11

CH 12: Modules over a PID

Let R denote a commutative ring with $1 \neq 0$

Let V denote an R-module.

Recall that V is <u>finitely generated</u> (fg)

whenever $\exists$ finite subset $S \subseteq V$ such that

$$V = \sum_{v \in S} Rv$$

Assume V is fg.

<u>Next general goal</u>: For R a PID, show that each R-submodule of V is fg.

To prove this cleanly, we introduce the notion of a Noetherian R-module.

DEF 1 Given an R-module V.

Call V <u>Noetherian</u> whenever:

For each sequence $\{W_i\}_{i=0}^{\infty}$ of R-submodules for V such that

$$W_i \subseteq W_{i+1} \qquad \text{for } i \geq 0,$$

$\exists$ $N \geq 0$ such that

$$W_i = W_{i+1} \qquad \text{for } i \geq N$$

LEM 2 Given an R-module V

TFAE

(i) V is Noetherian

(ii) each R-submodule of V is fg.

pf (i) $\to$ (ii) Given an R-submodule W of V, show W is fg. Suppose not.

Pick $0 \neq w_1 \in W$

$$W \neq Rw_1 \quad \text{since } W \text{ not fg}$$

So $\exists \; w_2 \in W \setminus Rw_1$

$$W \neq Rw_1 + Rw_2 \quad \text{since } W \text{ not fg}$$

So $\exists \; w_3 \in W \setminus (Rw_1 + Rw_2)$

etc

this gives a sequence $\{w_i\}_{i=1}^{\infty}$ of elements in W

For $i \geq 0$ define

$$W_i = Rw_1 + \cdots + Rw_i$$

Obs

W_i is an R-submodule of V

By constr

$$W_i \subsetneq W_{i+1} \qquad i \geq 0$$

this contradicts the assumption that V is Noetherian.

So W is fg.

$(ii) \to (i)$ Given a sequence

$$\{W_i\}_{i=0}^{\infty} \qquad ☆$$

of R-submodules for V st

$$W_i \leq W_{i+1} \qquad \text{for } i \geq 0$$

Define
$$W = \bigcup_{i=0}^{\infty} W_i$$

One checks
W is an R-submodule of V

So
W has a finite gen set S

Each element of S is in some term of ☆

So $\exists\, N \geq 0$ such that $S \subseteq W_N$

Now $W = \sum_{v \in S} Rv \subseteq W_N \subseteq W$

So $W = V$. Now $W_i = W_{i+1}$ for $i \geq N$

$\square$

LEM 3 Given a Noetherian R-module V.
Then each R-submodule of V is Noetherian.

pf By Def 1. $\square$

LEM 4 Given a surjective R-module hom

$$\varphi : V \to V'$$

Assume V is Noetherian. Then V' is Noetherian.

pf Given a sequence $\{W_i'\}_{i=0}^{\infty}$ of R-submodules for V' such that

$$W_i' \le W_{i+1}' \qquad \text{for } i \ge 0$$

For $i \ge 0$ Define

$$W_i = \text{preimage of } W_i' \text{ under } \varphi$$
$$= \{ v \in V \mid \varphi(v) \in W_i' \}$$

Then

W_i is an R-submodule of V

and $$W_i \le W_{i+1} \qquad i \ge 0$$

So $\exists\ N \ge 0$ such that

$$W_i = W_{i+1} \qquad \text{for } i \ge N$$

Now for $i \ge N$,

$$W_i' = \varphi(W_i) = \varphi(W_{i+1}) = W_{i+1}'$$

$\square$

LEM 5 Given an R-module V and an R-submodule W.

Assume W and V/W are Noeth.

Then V is Noeth.

pf Given R-submodules of V:

$$W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots$$

Consider R-submodules of W:

$$W_0 \cap W \subseteq W_1 \cap W \subseteq W_2 \cap W \subseteq \cdots$$

$\exists$ $n \geq 0$ st

$$W_i \cap W = W_n \cap W \qquad \text{for } i \geq n.$$

Consider quot map

$$\varphi: \quad V \to V/W$$

Consider R-submodules of V/W:

$$\varphi(W_0) \subseteq \varphi(W_1) \subseteq \varphi(W_2) \subseteq \cdots$$

$\exists \quad m \geq 0 \quad$ st

$$\varphi(W_i) = \varphi(W_{i+1}) \qquad \text{fr } i \geq m$$

Define

$$N = \max(n, m)$$

For $i \geq N$ show

$$W_i = W_{i+1}$$

Given $w \in W_{i+1}$. show $w \in W_i$

Obs

$$\varphi(w) \in \varphi(W_{i+1}) = \varphi(W_i)$$

so

$$\exists \; w' \in W_i$$

st

$$\varphi(w) = \varphi(w')$$

so $\quad \varphi(w - w') = 0$

so $\quad w - w' \in W$

Also $\quad w \in W_{i+1} \qquad w' \in W_i \subseteq W_{i+1}$

so $\quad w - w' \in W_{i+1}$

So

$$w - w' \in W_{i+1} \cap W = W_i \cap W \subseteq W_i$$

Now

$$w = \underset{\substack{\cap \\ W_i}}{w'} + \underset{\substack{\cap \\ W_i}}{\underbrace{w - w'}}$$

$$\in W_i$$

$\square$

COR 6 Given Noetherian R-modules U, V.

Then the direct product R-module $U \times V$ is Noetherian.

pf Recall the map

$$\begin{aligned} U &\to U \times V \\ a &\to (a, 0) \end{aligned}$$

is an injective R-module hom. Call the image W

W is an R-submodule of $U \times V$

The map

$$\begin{aligned} U \times V &\to V \\ (a, b) &\to b \end{aligned}$$

is a surjective R-module hom with kernel W

The R-modules U, W are iso so W is Noeth.

The R-modules V, $\frac{U \times V}{W}$ are iso so $\frac{U \times V}{W}$ is Noeth.

Now $U \times V$ is Noetherian by LEM 5. $\square$

LEM 7. Assume R is a PID.

Then the R-module R is Noetherian.

pf The R-submodules of R are the ideals of R

Each ideal in R is principle

So each R-submodule of R is generated by a single element.

Now R-module R is Noetherian by LEM 2. $\square$

Thm 8 Assume R is a PID

Given a fg R-module V.

Then V is Noetherian.

pf $\exists$ finite subset $S \subseteq V$ that generates V

Write $S = \{ v_1, v_2, \ldots, v_n \}$

Consider the free R-module

$$R^n = \underbrace{R \times R \times \cdots \times R}_{n}$$

The map

$$R^n \longrightarrow V$$

$$(a_1, a_2, \ldots, a_n) \longrightarrow a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

is a surjective R-module hom.

To show V is Noetherian, by LEM 4 it suffices to show R^n is Noetherian.

R^n is Noetherian by Cor 6 and LEM 7. $\square$

We will need the following fact.

LEM 9 Assume R is an integral domain.

Given a free R-module of finite rank n:

$$V = R^n$$

Given $m > n$ and

$$v_i \in V \qquad 1 \le i \le m$$

Then $\exists\ c_1, c_2, \dots, c_m \in R$, not all 0, such that

$$\sum_{i=1}^{m} c_i v_i = 0$$

pf View

$$R^n = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \;\middle|\; a_i \in R,\ 1 \le i \le n \right\}$$

Define $A \in \mathrm{Mat}_m(R)$ by

$$A = \begin{array}{c} \begin{pmatrix} v_1 & v_2 & \cdots & v_m \\ \hline & 0 & & \end{pmatrix} \begin{array}{l} \updownarrow n \\ \updownarrow m-n \end{array} \\ \longleftarrow m \longrightarrow \end{array}$$

A has a zero row so

$$\det(A) = 0$$

Back in Ch 11, we saw $\exists\ 0 \neq c \in \mathbb{R}^m$ st

$$Ac = 0$$

Write

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \qquad c_i \in \mathbb{R}$$

c_i not all 0

By matrix mult

$$\sum_{i=1}^{m} c_i v_i = 0$$

$\square$