Lecture 31 Friday April 8

$R$ = commutative ring with $1 \neq 0$

Thm 16 (Cramers rule) For $A \in \mathrm{Mat}_n(R)$,

for $$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in R^n$$

consider equations in the unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} A_{11} x_1 + A_{12} x_2 + \cdots + A_{1n} x_n &= c_1, \\ A_{21} x_1 + A_{22} x_2 + \cdots + A_{2n} x_n &= c_2, \\ &\cdots \\ A_{n1} x_1 + A_{n2} x_2 + \cdots + A_{nn} x_n &= c_n. \end{aligned}$$

Then for $1 \le j \le n$,

$$\det(A)\, x_j = \det\left(\text{matrix obtained from } A \text{ by replacing col}_j \text{ by } c\right)$$

Pf By construction

$$C = \sum_{i=1}^{n} \underset{\substack{\uparrow \\ \text{col } i \text{ of } A}}{A_i}\, x_i$$

So

$$\det(A_1, \dots, A_{j-1}, C, A_{j+1}, \dots, A_n)$$

$$= \sum_{i=1}^{n} x_i \underbrace{\det(A_1, \dots, A_{j-1}, A_i, A_{j+1}, \dots, A_n)}_{\substack{\shortparallel \\ 0 \text{ unless } i=j \text{ by LEM5}}}$$

$$= x_j \det(\underbrace{A_1, \dots, A_{j-1}, A_j, A_{j+1}, \dots, A_n}_{A})$$

$$= x_j \det(A)$$

□

COR 17 Assume $R$ is an integral domain.

Given $A \in Mat_n(R)$

TFAE

(i) $\exists$ nmo $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in R^n$ st

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0$$

(ii) $\det(A) = 0$

pf (i) $\to$ (ii) We have

$$\sum_{i=1}^{n} \underset{\substack{\uparrow \\ \text{col } i \text{ of } A}}{A_i} x_i = 0$$

Apply Cramer's rule

For $1 \le j \le n$

$$\det(A)\ x_j = \det(\text{matrix with col } j \text{ equal } 0)$$

$$= 0$$

$x_j$ not all $0$ so

$$\det(A) = 0$$

(ii) → (i) Embed $R$ into its field of fractions $F$

View $A \in Mat_n(F)$

By lin alg $\exists$ nonzero

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in F^n$$

st

$$A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = 0$$

Put $y_i$ over common denom:

$\exists\ 0 \neq r \in R$ s.t.

$r y_i \in R \qquad 1 \le i \le n$

For $1 \le i \le n$ $\quad y_i = 0$ iff $r y_i = 0$

We have

$$0 = r A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= A \begin{pmatrix} r y_1 \\ r y_2 \\ \vdots \\ r y_n \end{pmatrix}$$

Take $x_i = r y_i$ for $1 \le i \le n$

$\square$

Given $A \in \mathrm{Mat}_n(R)$

Define the <u>cofactor matrix of $A$</u> to have $(i,j)$-entry the $(i,j)$-cofactor of $A$, for $1 \le i,j \le n$.

Let $B$ = transpose of the cofactor matrix for $A$

Thm 18 With the above notation,

(i) $AB = \det(A)\, I$,

(ii) $BA = \det(A)\, I$.

Pf (i) For $1 \le i, j \le n$ compute the $(i,j)$-entry of $AB$ using matrix mult. This entry is the row $j$ cofactor expansion for the determinant of:

the matrix obtained from $A$ by replacing row $j$ by row $i$ $\quad *$

- For $i = j$, $*$ is just $A$ so its det is

$$\det(A)$$

- For $i \neq j$, $*$ has rows $i$ and $j$ identical so its det is

$$0$$

Therefore

$$AB = \det(A) I$$

(ii) For $1 \le i, j \le n$ compute the $(i,j)$-entry of $BA$ using matrix mult. This entry is the column $i$ cofactor expansion for the determinant of:

The matrix obtained from $A$ by replacing column $i$ by column $j$ $**$

- For $i = j$ $**$ is just $A$ so its det is

$$\det(A)$$

- For $i \neq j$, $**$ has columns $i$ and $j$ identical, so its det is

$$0$$

Therefore

$$BA = \det(A)\, I.$$

$\square$

Cor 19 For $A \in Mat_n(R)$ TFAE:

(i) $A$ is a unit in $Mat_n(R)$

(ii) $\det(A)$ is a unit in $R$

Assume (i), (ii) hold. Then

$$A^{-1} = \frac{\text{transpose of cofactor matrix for } A}{\det(A)} \quad = B$$

pf (i) → (ii) $\exists$ $C \in Mat_n(R)$ st

$$AC = I = CA$$

Take det:

$$\det(A)\cdot\det(C) = \underset{\underset{1}{\shortparallel}}{\det(I)} = \det(C)\det(A)$$

So $\det(C)$ is the inverse of $\det(A)$

(ii) → (i) $\exists$ $r \in R$ st

$$\det(A)\, r = 1 = r \det(A)$$

By Thm 18 $AB = \det(A) I = BA$

So $A(Br) = I = (Br)A$ So $A^{-1} = Br$

$\square$

## Example

$$A \in Mat_3(\mathbb{R})$$

$$A = \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix}$$

$\det(A) =$

$$asz \quad + \quad btx \quad + \quad cry$$

$$- aty \quad - \quad brz \quad - \quad csx$$

Find the minors of A

$$\begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix} \quad \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix}$$

Matrix of minors for A

$$\begin{pmatrix} \begin{vmatrix} s & t \\ y & z \end{vmatrix} & \begin{vmatrix} r & t \\ x & z \end{vmatrix} & \begin{vmatrix} r & s \\ x & y \end{vmatrix} \\ \\ \begin{vmatrix} b & c \\ y & z \end{vmatrix} & \begin{vmatrix} a & c \\ x & z \end{vmatrix} & \begin{vmatrix} a & b \\ x & y \end{vmatrix} \\ \\ \begin{vmatrix} b & c \\ s & t \end{vmatrix} & \begin{vmatrix} a & c \\ r & t \end{vmatrix} & \begin{vmatrix} a & b \\ r & s \end{vmatrix} \end{pmatrix}$$

$|M|$ means $\det(M)$

Matrix of cofactors for A

$$\begin{pmatrix} \begin{vmatrix} s & t \\ y & z \end{vmatrix} & -\begin{vmatrix} r & t \\ x & z \end{vmatrix} & \begin{vmatrix} r & s \\ x & y \end{vmatrix} \\ -\begin{vmatrix} b & c \\ y & z \end{vmatrix} & \begin{vmatrix} a & c \\ x & z \end{vmatrix} & -\begin{vmatrix} a & b \\ x & y \end{vmatrix} \\ \begin{vmatrix} b & c \\ s & t \end{vmatrix} & -\begin{vmatrix} a & c \\ r & t \end{vmatrix} & \begin{vmatrix} a & b \\ r & s \end{vmatrix} \end{pmatrix}$$

<u>Notation</u> Write

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \text{row 1 cofactor decomp of det}$$

etc

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \text{col 1 cofactor decomp of det}$$

etc.

Row Cofactor decompositions of det(A)

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = a\begin{vmatrix} s & t \\ y & z \end{vmatrix} - b\begin{vmatrix} r & t \\ x & z \end{vmatrix} + c\begin{vmatrix} r & s \\ x & y \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = -r\begin{vmatrix} b & c \\ y & z \end{vmatrix} + s\begin{vmatrix} a & c \\ x & z \end{vmatrix} - t\begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = x\begin{vmatrix} b & c \\ s & t \end{vmatrix} - y\begin{vmatrix} a & c \\ r & t \end{vmatrix} + z\begin{vmatrix} a & b \\ r & s \end{vmatrix}$$

Col Cofactor decompositions of det(A)

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = a\begin{vmatrix} s & t \\ y & z \end{vmatrix} - r\begin{vmatrix} b & c \\ y & z \end{vmatrix} + x\begin{vmatrix} b & c \\ s & t \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = -b\begin{vmatrix} r & t \\ x & z \end{vmatrix} + s\begin{vmatrix} a & c \\ x & z \end{vmatrix} - y\begin{vmatrix} a & c \\ r & t \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = c\begin{vmatrix} r & s \\ x & y \end{vmatrix} - t\begin{vmatrix} a & b \\ x & y \end{vmatrix} + z\begin{vmatrix} a & b \\ r & s \end{vmatrix}$$

The transpose of the cofactor matrix for A

$$\begin{pmatrix} \begin{vmatrix} s & t \\ y & z \end{vmatrix} & -\begin{vmatrix} b & c \\ y & z \end{vmatrix} & \begin{vmatrix} b & c \\ s & t \end{vmatrix} \\ -\begin{vmatrix} r & t \\ x & z \end{vmatrix} & \begin{vmatrix} a & c \\ x & z \end{vmatrix} & -\begin{vmatrix} a & c \\ r & t \end{vmatrix} \\ \begin{vmatrix} r & s \\ x & y \end{vmatrix} & -\begin{vmatrix} a & b \\ x & y \end{vmatrix} & \begin{vmatrix} a & b \\ r & s \end{vmatrix} \end{pmatrix}$$

$$A\left(\text{transpose of cofactor matrix for } A\right) =$$

$$\begin{pmatrix}
\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \det(A) &
\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0 &
\begin{vmatrix} a & b & c \\ r & s & t \\ a & b & c \end{vmatrix} = 0 \\[2em]
\begin{vmatrix} r & s & t \\ r & s & t \\ x & y & z \end{vmatrix} = 0 &
\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \det(A) &
\begin{vmatrix} a & b & c \\ r & s & t \\ r & s & t \end{vmatrix} = 0 \\[2em]
\begin{vmatrix} x & y & z \\ r & s & t \\ x & y & z \end{vmatrix} = 0 &
\begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix} = 0 &
\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = \det(A)
\end{pmatrix}$$

$$= \det(A)\, I$$

$$\left(\text{transpose of cofactor matrix for } A\right) A =$$

$$\begin{pmatrix}
\underset{\displaystyle \det(A)}{\underset{\shortparallel}{\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}}} & \underset{\displaystyle 0}{\underset{\shortparallel}{\begin{vmatrix} b & b & c \\ s & s & t \\ y & y & z \end{vmatrix}}} & \underset{\displaystyle 0}{\underset{\shortparallel}{\begin{vmatrix} c & b & c \\ t & s & t \\ z & y & z \end{vmatrix}}} \\[2em]
\underset{\displaystyle 0}{\underset{\shortparallel}{\begin{vmatrix} a & a & c \\ r & r & t \\ x & x & z \end{vmatrix}}} & \underset{\displaystyle \det(A)}{\underset{\shortparallel}{\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}}} & \underset{\displaystyle 0}{\underset{\shortparallel}{\begin{vmatrix} a & c & c \\ r & t & t \\ x & z & z \end{vmatrix}}} \\[2em]
\underset{\displaystyle 0}{\underset{\shortparallel}{\begin{vmatrix} a & b & a \\ r & s & r \\ x & y & x \end{vmatrix}}} & \underset{\displaystyle 0}{\underset{\shortparallel}{\begin{vmatrix} a & b & b \\ r & s & s \\ x & y & y \end{vmatrix}}} & \underset{\displaystyle \det(A)}{\underset{\shortparallel}{\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}}}
\end{pmatrix}$$

$$= \det(A)\ I$$