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Lecture 31 Friday April 8

 $R =$ commutative ring with $1 \neq 0$ Thm 16 (Cramers rule) For $A \in \text{Mat}_n(R)$,

$$\text{for } C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in R^n$$

consider equations in the unknowns x_1, x_2, \dots, x_n :

$$A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = c_1,$$

$$A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n = c_2,$$

...

$$A_{n1}x_1 + A_{n2}x_2 + \dots + A_{nn}x_n = c_n.$$

Then for $1 \leq j \leq n$,

$$\det(A) x_j = \det \left(\text{matrix obtained from } A \text{ by replacing col } j \text{ by } C \right)$$

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Pf By construction

$$C = \sum_{i=1}^n A_i x_i$$

\uparrow
 col i of A

So

$$\det(A_1, \dots, A_{j-1}, C, A_{j+1}, \dots, A_n)$$

$$= \sum_{i=1}^n x_i \det(A_1, \dots, A_{j-1}, A_i, A_{j+1}, \dots, A_n)$$

\parallel
 0 unless $i=j$ by LEM5

$$= x_j \det(A_1, \dots, A_{j-1}, A_j, A_{j+1}, \dots, A_n)$$

$\underbrace{\hspace{15em}}_A$

$$= x_j \det(A)$$

□

COR 17 Assume R is an integral domain.

Given $A \in \text{Mat}_n(R)$

TFAE

(i) \exists nonzero $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in R^n$ s.t.

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0$$

(ii) $\det(A) = 0$

pf (i) \rightarrow (ii) We have

$$\sum_{i=1}^n A_i x_i = 0$$

\uparrow
 col of A

Apply Cramer's rule

For $1 \leq j \leq n$

$$\det(A) x_j = \det(\text{matrix with col } j \text{ equal } 0)$$

$$= 0$$

x_j not all 0 so

$$\det(A) = 0$$

(ii) \rightarrow (i) Embed R into its field of fractions F

view $A \in \text{Mat}_n(F)$

By lin alg \exists vno

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in F^n$$

st $A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = 0$

Put y_i over commn denom:

$$\exists 0 \neq r \in R \text{ st } ry_i \in R \quad 1 \leq i \leq n$$

$\forall a \in R$
 $y_i = 0 \iff ry_i = 0$

We have

$$\begin{aligned} 0 &= r A \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \\ &= A \begin{pmatrix} ry_1 \\ ry_2 \\ \vdots \\ ry_n \end{pmatrix} \end{aligned}$$

Take $x_i = ry_i$ for $1 \leq i \leq n$



Given $A \in \text{Mat}_n(\mathbb{R})$

Define the cofactor matrix of A to have (i,j) -entry

the (i,j) -cofactor of A , for $1 \leq i, j \leq n$.

Let $B =$ transpose of the cofactor matrix for A

Thm 18 With the above notation,

$$(i) \quad AB = \det(A) I,$$

$$(ii) \quad BA = \det(A) I.$$

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Pf (i) For $1 \leq i, j \leq n$ compute the (i, j) -entry
of AB using matrix mult. This entry is the row i
cofactor expansion for the determinant of:

the matrix obtained from A by replacing row j by row i *

• For $i=j$, * is just A so its det is

$$\det(A)$$

• For $i \neq j$, * has rows i and j identical so its det is

$$0$$

Therefore

$$AB = \det(A)I$$

(ii) For $1 \leq i, j \leq n$ compute the (i, j) -entry of BA

using matrix mult. This entry is the column i

cofactor expansion for the determinant of i

The matrix obtained from A by replacing column i by column j ~~**~~

• For $i=j$ ~~**~~ is just A so its det is $\det(A)$

• For $i \neq j$, ~~**~~ has columns i and j identical, so its det is 0

Therefore

$$BA = \det(A) I.$$

□

Cor 19 For $A \in \text{Mat}_n(R)$ TFAE:

(i) A is a unit in $\text{Mat}_n(R)$

(ii) $\det(A)$ is a unit in R

Assume (i), (ii) hold. Then

$$A^{-1} = \frac{\text{transpose of cofactor matrix for } A}{\det(A)} = B$$

pf (i) \rightarrow (ii) $\exists C \in \text{Mat}_n(R)$ st

$$AC = I = CA$$

Take det:

$$\det(A) \det(C) = \det(I) = \det(C) \det(A)$$

\downarrow
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So $\det(C)$ is the inverse of $\det(A)$

(ii) \rightarrow (i) $\exists r \in R$ st

$$\det(A) r = 1 = r \det(A)$$

By Prop 18

$$AB = \det(A) I = BA$$

so

$$A(Br) = I = (Br)A \quad \text{So } A^{-1} = Br$$

□

Example

$$A \in \text{Mat}_3(\mathbb{R})$$

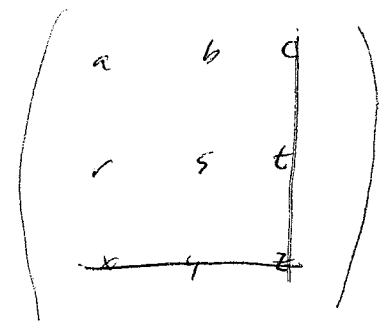
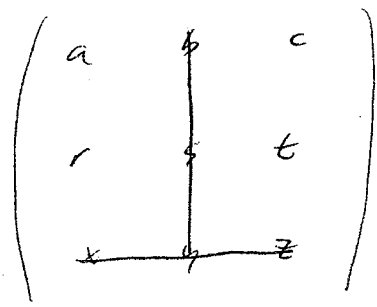
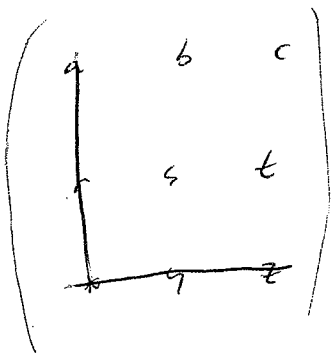
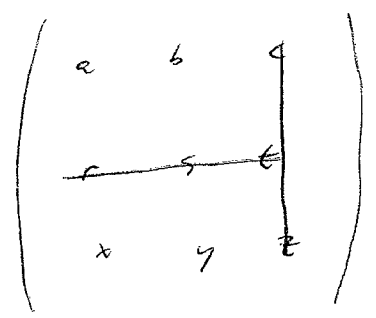
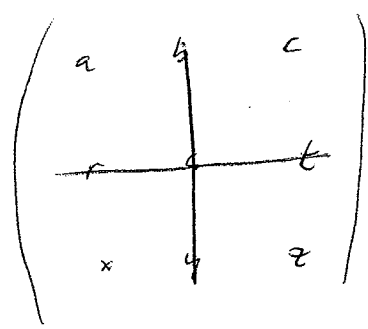
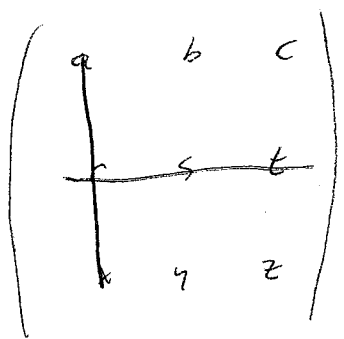
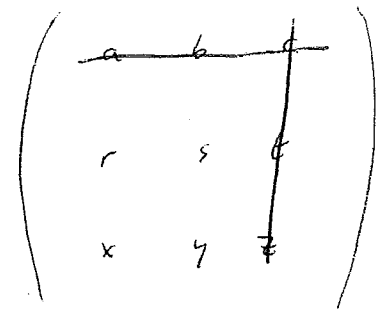
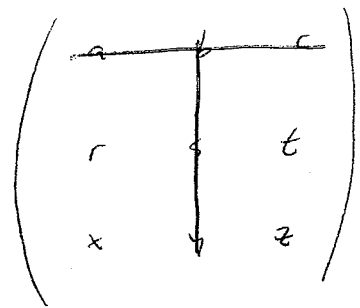
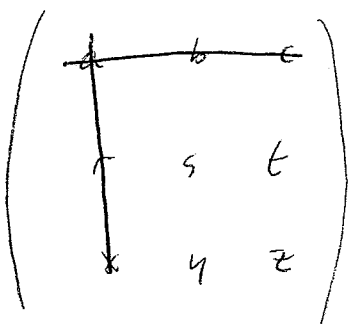
$$A = \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix}$$

$$\det(A) =$$

$$asz + btx + cry$$

$$-aty - brz - csx$$

Find the minors of A



Matrix of minors for A

$$\begin{vmatrix} s & t \\ y & z \end{vmatrix}$$

$$\begin{vmatrix} r & t \\ x & z \end{vmatrix}$$

$$\begin{vmatrix} r & s \\ x & y \end{vmatrix}$$

$$\begin{vmatrix} b & c \\ y & z \end{vmatrix}$$

$$\begin{vmatrix} a & c \\ x & z \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

$$\begin{vmatrix} b & c \\ s & t \end{vmatrix}$$

$$\begin{vmatrix} a & c \\ r & t \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ r & s \end{vmatrix}$$

$|M|$ means $\det(M)$

Matrix of cofactors for A

$$\begin{pmatrix} \begin{vmatrix} s & t \\ y & z \end{vmatrix} & - \begin{vmatrix} r & t \\ x & z \end{vmatrix} & \begin{vmatrix} r & s \\ x & y \end{vmatrix} \\ - \begin{vmatrix} b & c \\ y & z \end{vmatrix} & \begin{vmatrix} a & c \\ x & z \end{vmatrix} & - \begin{vmatrix} a & b \\ x & y \end{vmatrix} \\ \begin{vmatrix} b & c \\ s & t \end{vmatrix} & - \begin{vmatrix} a & c \\ r & t \end{vmatrix} & \begin{vmatrix} a & b \\ r & s \end{vmatrix} \end{pmatrix}$$

Notation

Write

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}$$

= row 1 cofactor decomp of det

etc

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}$$

= col 1 cofactor decomp of det

etc.

Row Cofactor decompositions of $\det(A)$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = a \begin{vmatrix} s & t \\ y & z \end{vmatrix} - b \begin{vmatrix} r & t \\ x & z \end{vmatrix} + c \begin{vmatrix} r & s \\ x & y \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = -r \begin{vmatrix} b & c \\ y & z \end{vmatrix} + s \begin{vmatrix} a & c \\ x & z \end{vmatrix} - t \begin{vmatrix} a & b \\ x & y \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = x \begin{vmatrix} b & c \\ s & t \end{vmatrix} - y \begin{vmatrix} a & c \\ r & t \end{vmatrix} + z \begin{vmatrix} a & b \\ r & s \end{vmatrix}$$

Col Cofactor decompositions of $\det(A)$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = a \begin{vmatrix} s & t \\ y & z \end{vmatrix} - r \begin{vmatrix} b & c \\ y & z \end{vmatrix} + x \begin{vmatrix} b & c \\ s & t \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = -b \begin{vmatrix} r & t \\ x & z \end{vmatrix} + a \begin{vmatrix} a & c \\ x & z \end{vmatrix} - y \begin{vmatrix} a & c \\ r & t \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix} = c \begin{vmatrix} r & s \\ x & y \end{vmatrix} - t \begin{vmatrix} a & b \\ x & y \end{vmatrix} + z \begin{vmatrix} a & b \\ r & s \end{vmatrix}$$

The transpose of the cofactor matrix of A

$$\begin{pmatrix} \begin{vmatrix} s & t \\ y & z \end{vmatrix} & - \begin{vmatrix} b & c \\ y & z \end{vmatrix} & \begin{vmatrix} b & c \\ s & t \end{vmatrix} \\ - \begin{vmatrix} r & t \\ x & z \end{vmatrix} & \begin{vmatrix} a & c \\ x & z \end{vmatrix} & - \begin{vmatrix} a & c \\ r & t \end{vmatrix} \\ \begin{vmatrix} r & s \\ x & y \end{vmatrix} & - \begin{vmatrix} a & b \\ x & y \end{vmatrix} & \begin{vmatrix} a & b \\ r & s \end{vmatrix} \end{pmatrix}$$

A (transpose of cofactor matrix of A)

=

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$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}$$

"
det(A)

$$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix}$$

"
0

$$\begin{vmatrix} a & b & c \\ r & s & t \\ a & b & c \end{vmatrix}$$

"
0

$$\begin{vmatrix} r & s & t \\ r & s & t \\ x & y & z \end{vmatrix}$$

"
0

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}$$

"
det(A)

$$\begin{vmatrix} a & b & c \\ r & s & t \\ r & s & t \end{vmatrix}$$

"
0

$$\begin{vmatrix} x & y & z \\ r & s & t \\ x & y & z \end{vmatrix}$$

"
0

$$\begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix}$$

"
0

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}$$

"
det(A)

= det(A) I

(transpose of cofactor matrix for A) A =

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}$$

" $\det(A)$

$$\begin{vmatrix} b & b & c \\ s & s & t \\ y & y & z \end{vmatrix}$$

" 0

$$\begin{vmatrix} c & b & c \\ t & s & t \\ z & y & z \end{vmatrix}$$

" 0

$$\begin{vmatrix} a & a & c \\ r & r & t \\ x & x & z \end{vmatrix}$$

" 0

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}$$

" $\det(A)$

$$\begin{vmatrix} a & c & c \\ r & t & t \\ x & z & z \end{vmatrix}$$

" 0

$$\begin{vmatrix} a & b & a \\ r & s & r \\ x & y & x \end{vmatrix}$$

" 0

$$\begin{vmatrix} a & b & b \\ r & s & s \\ x & y & y \end{vmatrix}$$

" 0

$$\begin{vmatrix} a & b & c \\ r & s & t \\ x & y & z \end{vmatrix}$$

" $\det(A)$

= $\det(A) I$