Lecture 30 Wednesday April 6

R = commutative ring with $1 \neq 0$

Given R-modules V, W

Given an alternating multilinear function

$$\varphi : \underbrace{V \times V \times \cdots \times V}_{n} \longrightarrow W$$

For $v_1, v_2, \ldots, v_n \in V$ consider

$$\varphi(v_1, v_2, \ldots, v_n) \qquad *$$

LEM 9 Given $v_1', v_2', \dots, v_n' \in V$ that are related to $v_1, v_2, \dots, v_n$ by

$$v_j' = \sum_{i=1}^{n} a_{ij} v_i \qquad 1 \le j \le n$$

$$a_{ij} \in R$$

Then

$$\varphi(v_1', v_2', \dots, v_n') =$$

$$\sum_{\sigma \in S_n} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} \, \varepsilon(\sigma) \, \varphi(v_1, v_2, \dots, v_n)$$

pf In

$$\varphi(v_1', v_2', \dots, v_n')$$

eval each v_j' in terms of $v_1, v_2, \dots, v_n$:

$\varphi(v_1', v_2', \ldots, v_n') =$

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} a_{i_1 1} a_{i_2 2} \cdots a_{i_n n} \underbrace{\varphi(v_{i_1}, v_{i_2}, \ldots, v_{i_n})}$$

//

0 unless $i_1, i_2, \ldots, i_n$ are mutually distinct, ie a permutation of $1, 2, \ldots, n$

In this case write

$i_1, i_2, \ldots, i_n = \sigma(1), \sigma(2), \ldots, \sigma(n)$

for some $\sigma \in S_n$

$$= \sum_{\sigma \in S_n} a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} \underbrace{\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)})}_{\text{// LEM 7}}$$

$\varepsilon(\sigma)\, \varphi(v_1, v_2, \ldots, v_n)$

$\square$

We now formally define

$$\det : \mathrm{Mat}_n(R) \to R$$

Recall the free R-module

$$\underset{\underset{V}{\parallel}}{R^n} = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \;\middle|\; \alpha_i \in R \quad 1 \le i \le n \right\}$$

For $1 \le i \le n$ def

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} \leftarrow i \qquad \in V$$

View

$$\mathrm{Mat}_n(R) = \underbrace{V \times V \times \cdots \times V}_{n}$$

So identity matrix is

$$I = (e_1, e_2, \ldots, e_n)$$

Thm 10 $\exists$ unique alternating multilinear form

$$\det : \mathrm{Mat}_n(R) \to R$$

such that $\det(I) = 1$.

Moreover for $A \in \mathrm{Mat}_n(R)$,

$$\det(A) = \sum_{\sigma \in S_n} A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(n)n} \, \varepsilon(\sigma) \qquad \star$$

pf Existence:

For $A \in \mathrm{Mat}_n(R)$ define $\det(A)$ by $\star$.

By construction det is multilinear and $\det(I) = 1$.

One checks det is alt.

Uniqueness:

Given an alternating multilinear form

$$\varphi : \mathrm{Mat}_n(R) \to R$$

st $\varphi(I) = 1$.

Show $\varphi = \det$

For $A \in Mat_n(R)$ show

$$\varphi(A) = \det(A)$$

For $1 \le j \le n$ let

A_j = column j of A

So
$$A_j = \sum_{i=1}^{n} A_{ij} e_i$$

View
$$A = (A_1, A_2, \ldots, A_n) \in V \times V \times \cdots \times V$$

By LEM 9,

$$\varphi(A) = \varphi(A_1, A_2, \ldots, A_n)$$

$$= \sum_{\sigma \in S_n} A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(n)n} \, \varepsilon(\sigma) \underbrace{\varphi(e_1, e_2, \ldots, e_n)}_{= \varphi(I) = 1}$$

$$= \det(A)$$

$\square$

Prop 11 For $A \in \mathrm{Mat}_n(R)$ the transpose A^t satisfies

$$\det(A^t) = \det(A)$$

pf By Thm 10,

$$\det(A^t) = \sum_{\sigma \in S_n} \underbrace{(A^t)_{\sigma(1)1}}_{= A_{1\sigma(1)}} (A^t)_{\sigma(2)2} \cdots (A^t)_{\sigma(n)n} \, \varepsilon(\sigma) \quad \text{etc}$$

$$= \sum_{\sigma \in S_n} A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)} \, \varepsilon(\sigma)$$

$$\left[\; A_{\underbrace{j}_{=\sigma^{-1}(i)} \underbrace{\sigma(j)}_{= i}} \; \right]$$

± 1

$$= \sum_{\sigma \in S_n} A_{\sigma^{-1}(1)1} A_{\sigma^{-1}(2)2} \cdots A_{\sigma^{-1}(n)n} \, \underbrace{\varepsilon(\sigma)}_{= (\varepsilon(\sigma))^{-1} = \varepsilon(\sigma^{-1})}$$

[change vars $\sigma \to \sigma^{-1}$]

$$= \sum_{\sigma \in S_n} A_{\sigma(1)1} A_{\sigma(2)2} \cdots A_{\sigma(n)n} \, \varepsilon(\sigma)$$

$$= \det(A)$$

$\square$

Prop 12 For $A, B \in Mat_n(R)$,

$$\det(AB) = \det(A)\det(B)$$

pf For $1 \le j \le n$

Column j of AB is

$$(AB)_j = \sum_{i=1}^{n} A_i B_{ij}$$

Now by LEM 9,

$$\det(AB) = \det\left((AB)_1, (AB)_2, \ldots, (AB)_n\right)$$

$$= \underbrace{\sum_{\sigma \in S_n} B_{\sigma(1)1} B_{\sigma(2)2} \cdots B_{\sigma(n)n} \, \varepsilon(\sigma)}_{\det B} \underbrace{\det(A_1, A_2, \ldots, A_n)}_{\det(A)}$$

$$= \det B \det A$$

$$= \det A \det B \qquad \text{since } R \text{ is commutative}$$

$\square$

LEM 13 For

$A \in Mat_n(R)$, $B \in Mat_m(R)$,

$$\det\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = \det(A)\det(B)$$

pf obs

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & B \end{array}\right)$$

Take det of each side and use Prop 12 along with the formula ☆ in thm 10. $\square$

DEF 14 For $A \in Mat_n(R)$ and $1 \le i, j \le n$,

- the <u>(i,j)-minor of A</u> is the matrix in $Mat_{n-1}(R)$ obtained by deleting row i and col j of A
- the <u>(i,j)-cofactor of A</u> is $(-1)^{i+j}$ times the determinant of the (i,j)-minor of A

Thm 15 For $A \in Mat_n(R)$

(i) For $1 \leq j \leq n$

$$\det(A) = \sum_{i=1}^{n} A_{ij} \left((i,j)\text{-cofactor of } A \right)$$

"cofactor expansion down col j"

(ii) For $1 \leq i \leq n$

$$\det(A) = \sum_{j=1}^{n} A_{ij} \left((i,j)\text{-cofactor of } A \right)$$

"cofactor expansion along row i"

pf (i)

$$\det(A) = \det(A_1, A_2, \dots, \overset{\substack{\text{col } j\\ \downarrow}}{A_j}, \dots, A_n$$

$$A_j = \sum_{i=1}^{n} A_{ij} e_i$$

$$= \sum_{i=1}^{n} A_{ij} \det(A_1, A_2, \dots, \overset{\substack{\text{col } j\\ \downarrow}}{e_i}, \dots, A_n)$$

$$\begin{array}{cc} & j \\ i & \boxed{\begin{matrix} & 0 & \\ * & \vdots & * \\ & 0 & \\ *\cdots* & 1 & *\cdots* \\ & 0 & \\ * & \vdots & * \\ & 0 & \end{matrix}} \end{array}$$

LEM 8

$$= \sum_{i=1}^{n} A_{ij} \det(\underbrace{A_1 - A_{i1} e_i, A_2 - A_{i2} e_i, \dots, \overset{\substack{\text{col } j\\ \downarrow}}{e_i}, \dots A_n - A_{in} e_i}_{A_{ij}^{\vee}})$$

$$A_{ij}^{\vee} : \quad \begin{array}{cc} & j \\ i & \begin{pmatrix} & 0 & \\ * & \vdots & * \\ & 0 & \\ 0 \cdots 0 & 1 & 0 \cdots 0 \\ & 0 & \\ * & \vdots & * \\ & 0 & \end{pmatrix} \end{array}$$

To compute $\det(\check{A}_{ij})$

permute the cols of $\check{A}_{ij}$ to move col j to col 1

this changes the det by a factor of $(-1)^{j-1}$

Now permute the rows to send row i to row 1.

this changes the det by a factor of $(-1)^{i-1}$

Resulting matrix is

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & (i,j)\text{-minor} & \\ & & \text{of } A & \\ 0 & & & \end{array}\right)$$

So

$$\det(\check{A}_{ij}) = (-1)^{i+j}\left(\det \text{ of } (i,j)-\text{minor of } A\right)$$

$$= (i,j)-\text{cofactor of } A$$

(ii) Apply (i) to A^t and use Prop 11. $\square$