Lecture 29 Monday April 4

R = commutative ring with $1 \neq 0$

Given R-modules U, V, W

Given a function

$$\varphi : U \times V \to W$$

Recall φ is bilinear whenever

$$\varphi(u+u', v) = \varphi(u,v) + \varphi(u',v) \qquad u, u' \in U$$

$$\varphi(u, v+v') = \varphi(u,v) + \varphi(u,v') \qquad v, v' \in V$$

$$a\varphi(u,v) = \varphi(au, v) = \varphi(u, av) \qquad a \in R$$

Observe,

The function φ is bilinear if and only if both

(i) $\forall u \in U$ the map

$$\begin{aligned} V &\longrightarrow W \\ v &\longrightarrow \varphi(u,v) \end{aligned}$$

is an R-module homomorphism;

(ii) $\forall v \in V$ the map

$$\begin{aligned} U &\longrightarrow W \\ u &\longrightarrow \varphi(u,v) \end{aligned}$$

is an R-module homomorphism.

Special cases

condition	name for φ
$U = V$	bilinear function on V
$W = \mathbb{R}$	bilinear form

Ex Recall the determinant function

$$\det : \begin{array}{ccc} \mathrm{Mat}_2(R) & \to & R \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \to & ad-bc \end{array}$$

Recall the free R-module

$$\begin{array}{l} R^2 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in R \right\} \\ \shortparallel \\ V \end{array}$$

View

$$\mathrm{Mat}_2(R) = V \times V$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$$

Then

$$\det : V \times V \to R$$

is a bilinear form on V.

A property of det

For $v \in V$ write

$$v = \begin{pmatrix} a \\ c \end{pmatrix}$$

Obs

$$\det(v, v) = \det\begin{pmatrix} a & a \\ c & c \end{pmatrix} = ac - ac = 0$$

We now investigate this property.

LEM 1 Given R modules V, W

Given bilinear function

$$\varphi : V \times V \to W$$

st $\varphi(v, v) = 0 \quad \forall v \in V$

then $\varphi(u, v) = -\varphi(v, u) \quad \forall u, v \in V$

pf

$$\begin{aligned} 0 &= \varphi(u+v, u+v) \\ &= \underbrace{\varphi(u, u+v)}_{\varphi(u,u) + \varphi(u,v)} + \underbrace{\varphi(v, u+v)}_{\varphi(v,u) + \varphi(v,v)} \end{aligned}$$

$\varphi(u,u) = 0$, $\varphi(v,v) = 0$

$\square$

<u>Problem</u> For $V = \mathbb{R}^2$ describe all the bilinear forms

$$\varphi : V \times V \to \mathbb{R}$$

such that

$$\varphi(v,v) = 0 \qquad \forall v \in V$$

<u>Sol</u> Write

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $u, v \in V$ write

$$u = \begin{pmatrix} a \\ c \end{pmatrix} = ae_1 + ce_2$$

$$v = \begin{pmatrix} b \\ d \end{pmatrix} = be_1 + de_2$$

Then

$$\varphi(u,v) = \varphi(ae_1 + ce_2,\ be_1 + de_2)$$

$$= a\underbrace{\varphi(e_1,\ be_1+de_2)}_{\substack{\| \\ b\underbrace{\varphi(e_1,e_1)}_{\substack{\|\\0}} + d\varphi(e_1,e_2)}} + c\underbrace{\varphi(e_2,\ be_1+de_2)}_{\substack{\| \\ b\underbrace{\varphi(e_2,e_1)}_{\substack{\|\\-\varphi(e_1,e_2)}} + d\underbrace{\varphi(e_2,e_2)}_{\substack{\|\\0}}}}$$

$$= (ad-bc)\,\varphi(e_1,e_2)$$

$$= \det\begin{pmatrix} a & b \\ c & d\end{pmatrix}\underbrace{\varphi(e_1,e_2)}_{\substack{\|\\ \alpha}}$$

φ is α times the determinant function.

For $\alpha=1$, $\varphi = \det$ is unique sol.

We could define det to be this unique solution.

Next goal: for $n \geq 2$ define

$$\det : \mathrm{Mat}_n(R) \to R$$

using this approach.

DEF 2 Given R-modules

$$V_1, V_2, \ldots, V_n, \quad W$$

Given a function

$$\varphi: \quad V_1 \times V_2 \times \cdots \times V_n \to W$$

Call φ <u>multilinear</u> whenever for $1 \le i \le n$ and

$$v_j \in V_j \qquad 1 \le j \le n, \quad j \ne i,$$

the function

$$V_i \longrightarrow W$$
$$v \longrightarrow \varphi(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_n)$$

is an R-module homomorphism.

<u>Note</u> For $n=2$, multilinear is the same as bilinear.

Special cases

condition	name for φ
$V_1 = V_2 = \cdots = V_n = V$	multilinear function on V
$W = R$	multilinear form

DEF 3 Given R-modules V, W

Given a multilinear function

$$\varphi : \underbrace{V \times V \times \cdots \times V}_{n} \to W.$$

Call φ <u>alternating</u> whenever for $2 \le i \le n$,

$$\varphi(v_1, \ldots, v_{i-2}, v_i, v_i, v_{i+1}, \ldots, v_n) = 0$$

for all $v_1, \ldots, v_{i-2}, v, v_{i+1}, \ldots, v_n \in V$

— o —

Until further notice, assume φ is alternating.

For $v_1, v_2, \ldots, v_n \in V$ consider

$$\varphi(v_1, v_2, \ldots, v_n)$$

*

LEM 4 For $2 \le i \le n$, $*$ changes sign if we exchange v_{i-1} and v_i.

pf For $n=2$ this is just LEM 1.

For general n, apply LEM 1 to the bilinear function

$$V \times V \longrightarrow W$$
$$u, v \longrightarrow \varphi(v_1, \dots, v_{i-2}, u, v, v_{i+1}, \dots, v_n)$$

$\square$

LEM 5 For $1 \le i < j \le n$,

$$* = 0 \quad \text{if} \quad v_i = v_j$$

pf By induction on $j-i$.

Case $j-i=1$: By the definition of alternating

Case $j-i \geq 2$: Exchange v_i and v_{i+1}:

$$\varphi(v_1, \ldots, v_i, v_{i+1}, \ldots, v_j, \ldots, v_n)$$

$$= -\varphi(v_1, \ldots, v_{i+1}, \underbrace{v_i, \ldots, v_j}_{\text{closer}}, \ldots, v_n)$$

$$= 0 \quad \text{by induction.}$$

$\square$

LEM 6 For $1 \le i < j \le n$, $*$ changes sign if we exchange v_i, v_j.

pf By LEM 5 the bilinear function

$$V \times V \longrightarrow W$$
$$u, v \longrightarrow \varphi(v_1, \dots, v_{i-1}, u, v_{i+1}, \dots, v_{j-1}, v, v_{j+1}, \dots, v_n)$$

satisfies the conditions of LEM 1.

Result follows by LEM 1. $\square$

We now generalize LEM 6

(Aside) Consider the set

$$\{1, 2, \ldots, n\}$$

Recall the symmetric group S_n consists of the permutations of $\{1, 2, \ldots, n\}$

The group operation is composition.

For $1 \le i < n$ define $T_i \in S_n$ that sends "ith transposition"

$$i \leftrightarrow i+1$$

$$j \to j \qquad 1 \le j \le n, \quad j \ne i, \ j \ne i+1$$

The group S_n is generated by $T_1, T_2, \ldots, T_{n-1}$

Recall from 541, $\exists$ unique surjective group homomorphism

$$\varepsilon : S_n \longrightarrow \{\pm 1\}$$
$$\sigma \longrightarrow \varepsilon(\sigma)$$

called the sign function.

We have

$$\varepsilon(rs) = \varepsilon(r)\,\varepsilon(s) \qquad r, s \in S_n$$

$$\varepsilon(T_i) = -1 \qquad 1 \le i < n$$

— o —

Return to $*$

LEM 7 For $\sigma \in S_n$

$$\varphi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}) = \varepsilon(\sigma)\, \varphi(v_1, v_2, \ldots, v_n).$$

pf Write σ as a product of transpositions and use LEM 4. $\square$

LEM 8 For distinct i, j $(1 \le i, j \le n)$ and $a \in R$,

$*$ remains unchanged if we replace v_i by $v_i + a v_j$.

pf

$$\varphi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) \stackrel{?}{=} \varphi(v_1, \ldots, v_i + a v_j, \ldots, v_j, \ldots, v_n)$$

$$\|$$

$$\varphi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) + a \underbrace{\varphi(v_1, \ldots, \overset{\text{dup}}{v_j}, \ldots, \overset{\text{dup}}{v_j}, \ldots, v_n)}_{= 0}$$

ok

□