Lecture 28 Friday April 1

Given vector spaces V, W over F

Given a lin trans

$$\varphi : V \to W$$

We next construct a lin trans

$$\varphi^* : W^* \to V^*$$

called the transpose of φ.

For $f \in W^*$ define a function

$$\varphi^* f : \quad \begin{array}{l} V \to F \\ v \to f(\varphi(v)) \end{array}$$

LEM 9. The above function $\varphi^* f$ is a lin trans.

In other words

$$\varphi^* f \in V^*$$

pf. $\varphi^* f$ is the composition of linear trans

$$V \xrightarrow{\varphi} W \xrightarrow{f} F$$

$\square$

LEM 10 The map

$$\varphi^*: W^* \longrightarrow V^*$$
$$f \longrightarrow \varphi^* f$$

"the transpose of φ"

is a lin trans.

pf Check

$$\varphi^*(f+g) \stackrel{?}{=} \varphi^*(f) + \varphi^*(g) \qquad f, g \in W^*$$

Apply both sides to $v \in V$

$$(\varphi^*(f+g))(v) \stackrel{?}{=} (\varphi^*(f) + \varphi^*(g))(v)$$

$$= (f+g)(\varphi(v)) = f(\varphi(v)) + g(\varphi(v))$$

$$(\varphi^*(f) + \varphi^*(g))(v) = \varphi^*(f)(v) + \varphi^*(g)(v) = f(\varphi(v)) + g(\varphi(v))$$

ok

$$\varphi^*(\alpha f) \stackrel{?}{=} \alpha \varphi^*(f) \qquad \alpha \in F \quad f \in W^*$$

Apply both sides to $v \in V$

$$(\varphi^*(\alpha f))(v) \stackrel{?}{=} (\alpha \varphi^*(f))(v)$$

$$(\varphi^*(\alpha f))(v) = (\alpha f)(\varphi(v)) = \alpha f(\varphi(v))$$

$$(\alpha \varphi^*(f))(v) = \alpha \underbrace{(\varphi^*(f)(v))}_{f(\varphi(v))}$$

ok.

$\square$

Given vector spaces V, W over F

Given lin trans $\varphi : V \to W$.

Get transpose lin trans $\varphi^* : W^* \to V^*$

We now explain why φ^* is called the transpose of φ

Assume $\dim V < \infty$, $\dim W < \infty$

Given

basis $\{v_i\}_{i=1}^n$ for V

Get dual basis $\{v_i^*\}_{i=1}^n$ for V^*

Given basis $\{w_i\}_{i=1}^m$ for W

Get dualbasis $\{w_i^*\}_{i=1}^m$ for W^*

LEM 11 With above notation,

the following matrices are transpose:

(i) the matrix rep φ wrt $\{v_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^m$ $(=A)$

(ii) the matrix rep φ^* wrt $\{w_i^*\}_{i=1}^m$ and $\{v_i^*\}_{i=1}^n$ $(=B)$

pf We have

$$\varphi(v_j) = \sum_{i=1}^{m} A_{ij} w_i \qquad 1 \le j \le n$$

$$\varphi^*(w_j^*) = \sum_{i=1}^{n} \underset{A_{ji}}{\overset{\beta_{ij}}{\shortparallel ?}} v_i^* \qquad 1 \le j \le m$$

show

$$\varphi^*(w_j^*) \overset{?}{=} \sum_{i=1}^{n} A_{ji} v_i^* \qquad 1 \le j \le m$$

Apply both sides to $v \in V$

WLOG $v = v_r$ $(1 \le r \le n)$

show

$$(\varphi^*(w_j^*))(v_r) \overset{?}{=} \left(\sum_{i=1}^{n} A_{ji} v_i^*\right)(v_r)$$

$$(\varphi^*(w_j^*))(v_r) = w_j^*(\varphi(v_r)), \quad \varphi(v_r) = \sum_{i=1}^{m} A_{ir} w_i, \quad w_j^*\left(\sum_{i=1}^{m} A_{ir} w_i\right) = A_{jr}$$

$$\left(\sum_{i=1}^{n} A_{ji} v_i^*\right)(v_r) = \sum_{i=1}^{n} A_{ji} v_i^*(v_r) = A_{jr}$$

ok

□

Prop 12 Given fd vector spaces V, W over F

Given lin trans $\varphi : V \to W$.

Then

(i) $(\ker(\varphi))^{\perp} = \varphi^*(W^*)$

(ii) $(\varphi(V))^{\perp} = \ker(\varphi^*)$

pf show (★)

$$(\ker(\varphi))^{\perp} \supseteq \varphi^*(W^*)$$

For $f \in W^*$ show

$$\varphi^*(f) \overset{?}{\in} (\ker(\varphi))^{\perp}$$

For $v \in \ker(\varphi)$ show

$$(\varphi^*(f))(v) \overset{?}{=} 0$$

$$\| $$

$$f(\underbrace{\varphi(v)}_{\substack{\| \\ 0}}) \quad \text{ok}$$

$$\underbrace{\qquad}_{\substack{\| \\ 0}}$$

Show

$$\varphi(V)^{\perp} \supseteq \ker(\varphi^*) \qquad (\star\star)$$

For $f \in \ker(\varphi^*)$ show

$$f \overset{?}{\in} (\varphi(V))^{\perp}$$

For $v \in V$ show

$$f(\varphi(v)) \overset{?}{=} 0$$

$$f(\varphi(v)) = (\underbrace{\underbrace{\varphi^*(f)}_{=0}(v)}_{=0}) = 0$$

Show equality use $(\star)$, $(\star\star)$

By (☆).

$$\dim(\varphi^*(W^*)) \leq \dim(\ker(\varphi))^\perp \qquad (1)$$
$$= \dim V - \dim(\ker(\varphi))$$

By (☆☆)

$$\dim(\ker(\varphi^*)) \leq \dim(\varphi(V)^\perp) \qquad (2)$$
$$= \dim W - \dim(\varphi(V))$$

Also

$$\dim V = \dim(\ker(\varphi)) + \dim(\varphi(V)) \qquad (3)$$

$$\dim W^* = \dim(\ker(\varphi^*)) + \dim(\varphi^*(W^*)) \qquad (4)$$
$$\shortparallel$$
$$\dim W$$

Adding (1) – (4) get $0 \leq 0$.

Therefore equality holds in (1), (2).

So equality holds in ☆, ☆☆. $\square$

COR 13 Given fd vector spaces V, W over F.

Given lin trans $\varphi : V \to W$

Consider transpose

$$\varphi^* : W^* \to V^*$$

then

$$\dim(\varphi(V)) = \dim(\varphi^*(W^*))$$

pf obs

$$\dim(\varphi(V)) = \dim V - \dim(\ker(\varphi))$$

$$= \dim(\underbrace{(\ker(\varphi))^{\perp}}_{\varphi^*(W^*)})$$

$\square$

For $m, n \in \mathbb{N}$

Given $A \in Mat_{m \times n}(F)$

View rows of A as vectors in F^n

Def $Row(A) =$ subspace of F^n spanned by rows of A

"row space of A"

View cols of A as vectors in F^m

Def $Col(A) =$ subspace of F^m spanned by cols of A

"col space of A"

Prop 14 With above notation,

$Row(A)$, $Col(A)$ have the same dimension.

pf let V, W denote vector spaces over F with

$$\dim(V) = n, \qquad \dim(W) = m$$

Fix basis $\{v_i\}_{i=1}^n$ for V
basis $\{w_i\}_{i=1}^m$ for W

let A rep a lin trans

$$\varphi : V \to W$$

wrt $\{v_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^m$

So A^t reps φ^* rel dual bases $\{w_i^*\}_{i=1}^m$ and $\{v_i^*\}_{i=1}^n$

We have

$$\dim(\mathrm{Col}(A)) = \dim(\varphi(V))$$

|| by Cor 13

$$\dim(\underbrace{\mathrm{Col}(A^t)}_{= \mathrm{Row}(A)}) = \dim(\varphi^*(W^*))$$

□

11.4 Determinants

Until further notice

R is a commutative ring with $1 \neq 0$

Given R-modules U, V, W

Given a function

$$\varphi : U \times V \to W$$

Call φ <u>bilinear</u> whenever

$$\varphi(u+u', v) = \varphi(u,v) + \varphi(u',v) \qquad u, u' \in U,\ v \in V$$

$$\varphi(u, v+v') = \varphi(u,v) + \varphi(u,v') \qquad u \in U,\ v, v' \in V$$

$$a\varphi(u,v) = \varphi(au, v) = \varphi(u, av) \qquad a \in R,\ u \in U,\ v \in V$$