Lecture 27 Wednesday March 30

11.3 The dual of a vector space

Fix a vector space V over F (poss $\dim V = \infty$)

Define

$$V^* = \mathrm{Hom}_F(V, F)$$
$$= \{ f \mid f : V \to F \text{ is a lin trans} \}$$

V^* is a vector space over F "the dual of V"

Describe V^*

Ex 1 Assume $\underset{n}{\overset{}{\dim V}} < \infty$, $\dim V = n$

Fix a basis for V:

$$v_1, v_2, \dots, v_n$$

☆

For $1 \le i \le n$ $\exists$ unique $v_i^* \in V^*$ such that

$$v_i^*(v_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \qquad 1 \le j \le n$$

One checks that

$$v_1^*, v_2^*, \ldots, v_n^*$$

is a basis for V^* (said to be <u>dual</u> to ☆)

We have

$$\dim(V^*) = n = \dim(V)$$

So the vector spaces V, V^* are iso

For $f \in V^*$,

$$f = \sum_{i=1}^{n} f(v_i)\, v_i^*$$

For $v \in V$,

$$v = \sum_{i=0}^{n} v_i^*(v)\, v_i$$

□

Ex 2 Let x = indet.

View the polynomial ring $F[x]$ as a vector space over F

with basis

$$1, x, x^2, \cdots$$

Describe the dual space $(F[x])^*$.

Define

$F[[x]]$ = set of all formal sums

$$\sum_{i \in \mathbb{N}} \alpha_i x^i \qquad \alpha_i \in F$$

do not require that fin many α_i nonzero

$F[[x]]$ is a vector space over F

One checks the map

$$(F[x])^* \longrightarrow F[[x]]$$

$$f \rightarrow \sum_{i \in \mathbb{N}} f(x^i)\, x^i$$

is an iso of vector spaces. $\square$

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For our vs V now consider

$$(V^*)^*$$

For $v \in V$ define a map

$$\hat{v} : \quad V^* \to F$$
$$\qquad f \to f(v)$$

LEM 3 The above map $\hat{v}$ is a lin trans.

In other words $\hat{v} \in (V^*)^*$

pf
$$\hat{v}(f+g) \stackrel{?}{=} \hat{v}(f) + \hat{v}(g) \qquad f, g \in V^*$$
$$\hat{v}(f) = f(v), \quad \hat{v}(g) = g(v)$$
$$\hat{v}(f+g) = (f+g)(v) = f(v) + g(v)$$

ok

$$\hat{v}(\alpha f) \stackrel{?}{=} \alpha \underbrace{\hat{v}(f)}_{= f(v)} \qquad \alpha \in F, \quad f \in V^*$$
$$\hat{v}(\alpha f) = (\alpha f)(v) = \alpha f(v)$$

ok

$\square$

LEM 4 With above notation, the map

$$\wedge : \begin{array}{ccc} V & \longrightarrow & (V^*)^* \\ v & \longmapsto & \hat{v} \end{array}$$

is a linear trans

pf check

$$\widehat{v+w} \overset{?}{=} \hat{v} + \hat{w} \qquad v, w \in V$$

Apply both sides to $f \in V^*$

$$\widehat{v+w}\,(f) \overset{?}{=} (\hat{v} + \hat{w})(f)$$

$$\widehat{v+w}\,(f) = f(v+w) = f(v) + f(w)$$

$$(\hat{v} + \hat{w})(f) = \hat{v}(f) + \hat{w}(f) = f(v) + f(w)$$

ok

$$\widehat{\alpha v} \overset{?}{=} \alpha(\hat{v}) \qquad \alpha \in F, \; v \in V$$

$$\underbrace{\widehat{\alpha v}\,(f)}_{= f(\alpha v) = \alpha f(v)} \overset{?}{=} \alpha \underbrace{(\hat{v})(f)}_{= f(v)}$$

ok.

$\square$

LEM 5 Assume $\underset{n}{\overset{\shortparallel}{\dim V}} < \infty$

Then the map

$$\wedge : \begin{array}{ccc} V & \longrightarrow & (V^*)^* \\ v & \longmapsto & \hat{v} \end{array}$$

is an iso of vector spaces.

pf Obs $\dim V = \dim(V^*) = \dim((V^*)^*)$

Suf to show $\wedge$ is injective.

Given $v \in V$ st $\hat{v} = 0$

Show $v = 0$

Suppose $v \neq 0$.

Extend v to a basis for V:

$$v = v_1, v_2, \ldots, v_n$$

Consider the dual basis for V^*:

$$v_1^*, v_2^*, \ldots, v_n^*$$

Obs

$$0 = \hat{v}(v_1^*) = v_1^*(v) = v_1^*(v_1) = 1$$

cont.

So $v = 0$ $\square$

Assume $\underset{n}{\overset{\shortparallel}{\dim V}} < \infty$

Given subspace $U \subseteq V$

Define
$$U^\perp = \{ f \in V^* / \ f(u) = 0 \ \ \forall u \in U \}$$

$U^\perp$ = subspace of V^*

LEM 6 With above notation,
$$\dim U + \dim(U^\perp) = \dim V$$

pf Pick a basis for U:
$$v_1, v_2, \dots, v_r$$
Extend this to a basis for V:
$$v_1, v_2, \dots v_n$$
Consider dual basis for V^*:
$$v_1^*, v_2^*, \dots, v_n^*$$

For $f \in V^*$,
$$f = \sum_{i=1}^{n} f(v_i) v_i^*$$

So

$$f \in U^\perp \iff f(v_i) \text{ fr } 1 \le i \le r$$
$$\iff f \in \sum_{i=r+1}^{n} F v_i^*$$

Now $U^\perp$ has basis

$$v_{r+1}^*, \dots, v_n^*$$

So

$$\dim U^\perp = n - r$$
$$= \dim V - \dim U$$

$\square$

Assume $\dim V < \infty$

Given a subspace $W \subseteq V^*$

define
$$W^\perp = \{ v \in V \mid f(v) = 0 \quad \forall f \in W \}$$

$W^\perp$ is a subspace of V

LEM 7 With above notation,

$$\dim W + \dim(W^\perp) = \dim V$$

pf Similar to pf of LEM 6. $\square$

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LEM 8 Assume $\dim V < \infty$

For subspaces $U \subseteq V$ and $W \subseteq V^*$ TFAE

(i) $W = U^\perp$

(ii) $U = W^\perp$

pf (i) → (ii) Obs

$$U \subseteq (U^\perp)^\perp$$

Equality holds since U, $(U^\perp)^\perp$ have same dimension by LEM 6, 7.

(ii) → (i) Obs

$$W \subseteq (W^\perp)^\perp$$

Equality holds since W, $(W^\perp)^\perp$ have same dimension by LEM 6, 7. $\square$