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Lecture 26 Monday March 28

LEM 7 Given fd vectr space V over F

Given

basis $\{v_i\}_{i=1}^n$ for V $\qquad *$

basis $\{v_i'\}_{i=1}^n$ for V $\qquad *'$

Let

S = transition matrix fm $*$ to $*'$

Given lin trans

$$\varphi : V \to V$$

let

A = matrix rep φ rel $*$

A' = matrix rep φ rel $*'$

then

$$AS = SA'$$

In other words

$$A' = S^{-1}AS$$

pf In discussion above Lemma, take

$$W = V$$

$w_i = v_i \qquad 1 \le i \le n$

$w_i' = v_i' \qquad 1 \le i \le n$

So $S = T$

Now $AS = TA' = SA'$

$\square$

DEF 8 Given $A, A' \in \mathrm{Mat}_n(F)$

call A, A' <u>similar</u> whenever $\exists$ invertible

$S \in \mathrm{Mat}_n(F)$ st

$$A' = S^{-1} A S$$

Tensor products of vector spaces (concrete view)

Given fd vector spaces V, W

Given

basis $\{v_i\}_{i=1}^{n}$ for V

basis $\{w_j\}_{j=1}^{m}$ for W

Let $V \otimes W = V \otimes_F W$ denote the vectorspace

over F with basis

$$v_i \otimes w_j \qquad 1 \le i \le n, \qquad 1 \le j \le m \qquad *$$

So elements of $V \otimes W$ are formal linear combinations of $*$

Note

$$\dim(V \otimes W) = \dim(V) \times \dim(W)$$

For $v \in V$ and $w \in W$ define $v \otimes w$ as follows:

Write

$$v = \sum_{i=1}^{n} \alpha_i v_i \qquad \alpha_i \in F$$

$$w = \sum_{j=1}^{m} \beta_j w_j \qquad \beta_j \in F$$

then

$$v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \, v_i \otimes w_j$$

We have

$$(v+v')\otimes w = v\otimes w + v'\otimes w \qquad v, v' \in V$$

$$v\otimes(w+w') = v\otimes w + v\otimes w' \qquad w, w' \in W$$

$$\alpha(v\otimes w) = (\alpha v)\otimes w = v\otimes(\alpha w) \qquad \alpha \in F$$

LEM 9 With above notation,

Given basis $\{v_i'\}_{i=1}^n$ for V,

basis $\{w_i'\}_{i=1}^m$ for W.

Then the vectors

$$v_i'\otimes w_j' \qquad 1\le i\le n, \qquad 1\le j\le m \qquad **$$

form a basis for $V\otimes W$

pf Obs

$$\dim(V\otimes W) = nm = \#\text{ vectors in } **$$

Suf to show $**$ spans $V\otimes W$

For $v\in V$ and $w\in W$ write

$$v = \sum_{i=1}^n a_i v_i' \qquad a_i \in F$$

$$w = \sum_{j=1}^m b_j w_j' \qquad b_j \in F$$

Then
$$v\otimes w = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \, v_i'\otimes w_j'$$

$\square$

LEM 10 Given fd vector spaces

V, V', W, W' over F.

Given lin trans

$$\varphi : V \to V', \qquad \phi : W \to W'$$

then $\exists$ lin trans

$$\varphi \otimes \phi : V \otimes W \to V' \otimes W'$$

that sends

$$v \otimes w \longrightarrow \varphi(v) \otimes \phi(w)$$

for all $v \in V$ and $w \in W$.

pf Fix

basis $\{v_i\}_{i=1}^{n}$ for V

basis $\{w_j\}_{j=1}^{m}$ for W

So

$$v_i \otimes w_j \qquad 1 \le i \le n, \qquad 1 \le j \le m$$

is a basis for $V \otimes W$

$\exists$ lin trans

$$\varphi \otimes \phi : \quad V \otimes W \longrightarrow V' \otimes W'$$
$$v_i \otimes w_j \longrightarrow \varphi(v_i) \otimes \phi(w_j)$$

Given $v \in V$ and $w \in W$ show

$$(\varphi \otimes \phi)(v \otimes w) = \varphi(v) \otimes \phi(w)$$

write

$$v = \sum_{i=1}^{n} \alpha_i v_i \qquad \alpha_i \in F$$

$$w = \sum_{j=1}^{m} \beta_j w_j \qquad \beta_j \in F$$

So

$$v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \, v_i \otimes w_j$$

So

$$(\varphi \otimes \phi)(v \otimes w) = (\varphi \otimes \phi)\left(\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \, v_i \otimes w_j \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \underbrace{(\varphi \otimes \phi)(v_i \otimes w_j)}_{\varphi(v_i) \otimes \phi(w_j)}$$

$$= \underbrace{\left(\sum_{i=1}^{n} \alpha_i \varphi(v_i) \right)}_{\varphi(v)} \otimes \underbrace{\left(\sum_{j=1}^{m} \beta_j \phi(w_j) \right)}_{\phi(w)}$$

$$= \varphi(v) \otimes \phi(w)$$

$\square$

Consider lin trans

$\varphi : V \to V'$, $\qquad \phi : W \to W'$

as above.

Given

basis $\{v_i\}_{i=1}^{n}$ for V $\qquad *$

basis $\{v_i'\}_{i=1}^{N}$ for V' $\qquad *'$

basis $\{w_j\}_{j=1}^{m}$ for W $\qquad **$

basis $\{w_j'\}_{j=1}^{M}$ for W' $\qquad **'$

So $V \otimes W$ has basis

$v_i \otimes w_j \qquad 1 \le i \le n, \qquad 1 \le j \le m$ $\qquad ☆$

and $V' \otimes W'$ has basis

$v_i' \otimes w_j' \qquad 1 \le i \le N \qquad 1 \le j \le M$ $\qquad ☆'$

Find the matrix rep $\varphi \otimes \phi$ rel ☆ and ☆':

Let

A = matrix rep φ rel $*$, $*'$

B = matrix rep ϕ rel $**$, $**'$

So

$$\varphi(v_r) = \sum_{i=1}^{N} A_{ir} v_i' \qquad 1 \le r \le n$$

$$\phi(w_s) = \sum_{j=1}^{M} B_{js} w_j' \qquad 1 \le s \le m$$

Now

$$(\varphi \otimes \phi)(v_r \otimes w_s) = \varphi(v_r) \otimes \phi(w_s)$$

$$= \left(\sum_{i=1}^{N} A_{ir} v_i' \right) \otimes \left(\sum_{j=1}^{M} B_{js} w_j' \right)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{M} A_{ir} B_{js} (v_i' \otimes w_j')$$

This shows ...

For the matrix rep $\varphi \otimes \phi$ wrt $\mathcal{B}$ and $\mathcal{B}'$;

the rows are indexed by ordered pairs

(i, j) $\quad 1 \le i \le N, \quad 1 \le j \le M$

the cols are indexed by ordered pairs

(r, s) $\quad 1 \le r \le n, \quad 1 \le s \le m$

For row (i,j) and col (r,s) the corresp entry is

$A_{ir} B_{js}$

The above matrix is denoted $A \otimes B$

"kronecker product of A, B"

"tensor product of A, B"