Lecture 25 Friday March 18

Fix field F

Given fd vector spaces V, W over F

Given

basis $\{v_i\}_{i=1}^{n}$ for V $\quad *$

basis $\{w_i\}_{i=1}^{m}$ for W $\quad **$

For $\varphi \in Hom_F(V,W)$, consider

matrix A rep φ wrt $*$ and $**$

$$\varphi(v_j) = \sum_{i=1}^{m} A_{ij} v_i \qquad 1 \le j \le n$$

So $A \in Mat_{m \times n}(F)$.

The map

$$Hom_F(V,W) \longrightarrow Mat_{m\times n}(F)$$

$$\varphi \longrightarrow \text{matrix rep } \varphi \text{ wrt } * \text{ and } **$$

is iso of vector spaces.

COR 3 Given finite dim'l vector spaces V, W over F. Then the dimension of $\mathrm{Hom}_F(V, W)$ is

$$\dim(V) \times \dim(W).$$

pf By Prop 2. $\square$

3/18/16
3

Given three finite dim'l vector spaces over F:

U, V, W.

Given

basis $\{u_i\}_{i=1}^r$ for U $\quad *$

basis $\{v_i\}_{i=1}^s$ for V $\quad **$

basis $\{w_i\}_{i=1}^t$ for W $\quad ***$

Given lin transformations

$$U \xrightarrow[\varphi]{} V \xrightarrow[\phi]{} W$$

Consider

matrix rep φ rel $*, **$ $\quad (=A)$

matrix rep ϕ rel $**, ***$ $\quad (=B)$

matrix rep $\phi \circ \varphi$ rel $*, ***$ $\quad (=C)$

How are A, B, C related?

For $1 \le j \le r$

$$(\phi \circ \varphi)(u_j) = \sum_{i=1}^{t} C_{ij} w_i$$

Also

$$(\phi \circ \varphi)(u_j) = \phi(\varphi(u_j))$$

$$= \phi\left(\sum_{l=1}^{s} A_{lj} v_l\right)$$

$$= \sum_{l=1}^{s} A_{lj} \phi(v_l)$$

$$= \sum_{l=1}^{s} A_{lj} \left(\sum_{i=1}^{t} B_{il} w_i\right)$$

$$= \sum_{i=1}^{t} \left(\sum_{l=1}^{s} B_{il} A_{lj}\right) w_i$$

For $1 \le i \le t$ compare w_i - coefs to get

$$C_{ij} = \sum_{l=1}^{s} B_{il} A_{lj} \qquad \begin{array}{l} 1 \le i \le t \\ 1 \le j \le r \end{array}$$

In other words

$$C = BA$$

↑ matrix product

COR 4 The matrix product is associative.

pf Given matrices A, B, C

show

$$(AB)C = A(BC) \qquad *$$

[assume the dimensions are such that above products make sense]

View A, B, C as representing some lin trans φ, ϕ, ψ rel some given bases.

$$AB \quad \text{represents} \quad \varphi \circ \phi$$

$$(AB)C \quad \text{represents} \quad (\varphi \circ \phi) \circ \psi$$

$$BC \quad \text{..} \quad \phi \circ \psi$$

$$A(BC) \quad \text{..} \quad \varphi \circ (\phi \circ \psi)$$

But

$$(\varphi \circ \phi) \circ \psi = \varphi \circ (\phi \circ \psi)$$

So $*$ holds $\square$

For $n \geq 1$ define

$$Mat_n(F) = Mat_{n \times n}(F)$$

Matrix mult turns $Mat_n(F)$ into a ring with identity

$$I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \ 1 \end{pmatrix}$$

The map

$$\eta : \begin{array}{ccc} F & \to & Mat_n(F) \\ \alpha & \to & \alpha I \end{array}$$

turns $Mat_n(F)$ into an F-algebra.

Given vector space V over F with $\dim(V) = n$

Recall F-alg

$$End_F(V) = Hom_F(V, V)$$

Fix a basis $\{v_i\}_{i=1}^n$ for V $\ast$

For $\varphi \in End_F(V)$ $\exists$ unique $A \in Mat_n(F)$ st

$$\varphi(v_j) = \sum_{i=1}^n A_{ij} v_i \qquad 1 \leq j \leq n$$

Call A the <u>matrix that represents φ with respect to</u> $\ast$

<u>Obs</u> for $1 \in End_F(V)$

I = matrix that reps 1 rel $\ast$.

Prop 5 With above notation, the map

$$\varsigma : \begin{array}{ccc} End_F(V) & \to & Mat_n(F) \\ \varphi & \longrightarrow & \underbrace{\text{matrix rep } \varphi \text{ rel } *}_{= \varphi^{\varsigma}} \end{array}$$

is an F-algebra isomorphism

pf ς is vector space iso by Prop 2 (with $V=W$)

obs $1^{\varsigma} = I$

show ς respects mult

Fa $\varphi, \phi \in End_F(V)$

$$(\phi \circ \varphi)^{\varsigma} \overset{?}{=} \phi^{\varsigma} \varphi^{\varsigma}$$

this is just $C = BA$ from above COR 4. $\square$

Transition matrices

Given fd vector space V over F

Given

basis $\{u_i\}_{i=1}^n$ for V $\qquad *$

basis $\{v_i\}_{i=1}^n$ for V $\qquad **$

$\exists$ unique matrix $S \in Mat_n(F)$ st

$$v_j = \sum_{i=1}^n S_{ij} u_i \qquad 1 \le j \le n.$$

Call S the transition matrix from $*$ to $**$

Given

basis $\{w_i\}_{i=1}^n$ for V $\qquad ***$

Compare

trans matrix from $*$ to $***$ $\quad (=R)$

trans matrix from $*$ to $**$ $\quad (=S)$

trans matrix from $**$ to $***$ $\quad (=T)$

For $1 \le j \le n$,

$$w_j = \sum_{i=1}^{n} R_{ij} u_i$$

Also

$$w_j = \sum_{l=1}^{n} T_{lj} v_l$$

$$= \sum_{l=1}^{n} T_{lj} \left(\sum_{i=1}^{n} S_{il} u_i \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{l} S_{il} T_{lj} \right) u_i$$

For $1 \le i \le n$, compare u_i-coefs to get

$$R_{ij} = \sum_{l=1}^{n} S_{il} T_{lj} \qquad 1 \le i, j \le n$$

In other words

$$R = ST$$

LEM 6 Given fd vector space V over F

Given basis $\{u_i\}_{i=1}^n$ for V $*$

basis $\{v_i\}_{i=1}^n$ for V $**$

Let $S =$ transmatrix from $*$ to $**$.

Then S is invertible in $Mat_n(F)$

Moreover S^{-1} is the trans matrix from $**$ to $*$.

pf In the discussion above the lemma

take $w_i = u_i$ for $1 \le i \le n$

then $* = ***$ so

$$I = R$$
$$= ST$$

so $T = S^{-1}$

$\square$

Given fd vectr spaces V, W

Given

basis $\{v_i\}_{i=1}^n$ for V $\quad (*)$

basis $\{v_i'\}_{i=1}^n$ for V $\quad (*')$

basis $\{w_i\}_{i=1}^m$ for W $\quad (**)$

basis $\{w_i'\}_{i=1}^m$ for W $\quad (**')$

Given lin trans

$$\varphi : V \to W$$

Compare

matrix rep φ wrt $*$ and $**$ $\quad (= A)$

matrix rep φ wrt $*'$ and $**'$ $\quad (= A')$

Let

S = trans matrix fm $*$ to $*'$

T = trans matrix fm $**$ to $**'$

For $1 \le j \le n$,

$$\varphi(v_j) = \sum_{i=1}^{m} A_{ij} w_i$$

$$\varphi(v_j') = \sum_{s=1}^{m} A'_{sj} w'_s$$

$$\varphi\left(\sum_{r=1}^{n} S_{rj} v_r\right) \qquad\qquad \sum_{s=1}^{m} A'_{sj}\left(\sum_{i=1}^{m} T_{is} w_i\right)$$

$$\sum_{r=1}^{n} S_{rj}\, \varphi(v_r) \qquad\qquad \sum_{i=1}^{m}\left(\sum_{s=1}^{m} T_{is} A'_{sj}\right) w_i$$

$$\sum_{r=1}^{n} S_{rj}\left(\sum_{i=1}^{m} A_{ir} w_i\right)$$

$$\sum_{i=1}^{m}\left(\sum_{r=1}^{n} A_{ir} S_{rj}\right) w_i$$

So
$$\sum_{r=1}^{n} A_{ir} S_{rj} = \sum_{s=1}^{m} T_{is} A'_{sj} \qquad \begin{array}{l} 1 \le j \le n \\ 1 \le i \le m \end{array}$$

In other words

$$A S = T A'$$

or

$$A' = T^{-1} A S$$