Lecture 24 Monday March 14 3/14/16

1

Recall our vector space V over F

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ " F-module

Given subspace W of V

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ " F-submodule

Recall quotient

$$V/W = \text{vector space over } F$$

the quot map

$$\varphi : \begin{array}{ccc} V & \to & V/W \\ v & \to & v + W \end{array}$$

is a lin trans, ie F-module hom

φ is surj with $\ker(\varphi) = W$.

LEM 13 With above notation.

$$\dim(V) = \dim(W) + \dim(V/W)$$

pf First assume $\dim V < \infty$
$\overset{\shortparallel}{n}$

Given basis for V:

$v_1, v_2, \ldots, v_n$

Apply φ:

$\varphi(v_i) \quad 1 \le i \le n$

spans V/W

$\exists$ subset $S \subseteq \{1,2,\ldots,n\}$ st

$\varphi(v_i) \quad i \in S$

is basis for V/W.

Define $\hat{W} = \sum_{i \in S} F v_i$

obs $W \cap \hat{W} = 0$

Define $\bar{S} = \{1,2,\ldots,n\} \setminus S$

For $j \in \bar{S}$,

$$\varphi(v_j) = \sum_{i \in S} \alpha_{ij} \varphi(v_i) \qquad \alpha_{ij} \in F$$

So

$$\varphi\left(v_j - \sum_{i \in S} \alpha_{ij} v_i\right) = 0$$

So

$$v_j - \underbrace{\sum_{i \in S} \alpha_{ij} v_i}_{\substack{\parallel \text{ def} \\ \hat{v}_j \in \hat{W}}} \in W$$

Consider the vectors

$$v_j - \hat{v}_j \qquad j \in \bar{S} \qquad *$$

show $*$ is a basis for W

check $*$ is Lin indep:

Given $\displaystyle\sum_{j \in \bar{S}} \alpha_j (v_j - \hat{v}_j) = 0 \qquad \alpha_j \in F$

then $\displaystyle\sum_{j \in \bar{S}} \alpha_j v_j = \sum_{j \in \bar{S}} \alpha_j \hat{v} \in \hat{W} = \sum_{j \in S} F v_j$

So $\alpha_j = 0 \quad \forall j \in \bar{S}$ ✓

check $*$ spans W:

For $w \in W$

write $w = \sum_{i=1}^{n} \alpha_i v_i \qquad \alpha_i \in F$

$$w = \sum_{i \in S} \alpha_i v_i + \sum_{i \in \bar{S}} \alpha_i \underbrace{v_i}_{\hat{v}_i + v_i - \hat{v}_i}$$

$$\underbrace{w - \sum_{i \in \bar{S}} \alpha_i (v_i - \hat{v}_i)}_{\in W} = \underbrace{\sum_{i \in S} \alpha_i v_i}_{\in \hat{W}} + \underbrace{\sum_{i \in \bar{S}} \alpha_i \hat{v}_i}_{\in \hat{W}}$$

$W \cap \hat{W} = 0$

both sides 0

so

$$w = \sum_{i \in \bar{S}} \alpha_i (v_i - \hat{v}_i)$$

So $*$ is basis for W

$$\dim V \overset{?}{=} \dim W + \dim V/W$$

$$\begin{array}{ccccc} \dim V & \overset{?}{=} & \dim W & + & \dim V/W \\ \| & & \| & & \| \\ n & & |\bar{S}| & & |S| \end{array}$$

ok

Done for $\dim V < \infty$

Case $\dim V = \infty$ is similar.

$\square$

3/14/16

5

Given vector spaces V, W over F

Given lin trans $\varphi : V \to W$

Recall

$\ker(\varphi)$ is F-submodule of V

ie subspace of V

$\varphi(V)$ is F-submodule of W

ie subspace of W

LEM 14 With above notation

$$\dim V = \dim \ker(\varphi) + \dim \varphi(V)$$

pf Write

$$W = \ker(\varphi)$$

Recall $\exists$ F-module iso (ie vs iso)

$$V/W \longrightarrow \varphi(V)$$

So

$$\dim V/W = \dim \varphi(V)$$

Result follows by LEM 13

$\square$

LEM 15 Given vector spaces V, W over F with the same finite dimension.

Given lin trans $\varphi : V \to W$

TFAE

(i) φ is a bijection

(ii) φ is injective

(iii) φ is surjective

Call φ "invertible" or "nonsingular" whenever (i)–(iii) hold.

pf Use LEM 14 $\square$

LEM 16 Given vector spaces V, W over F with the same finite dimension.

Given lin trans $\varphi: V \to W$.

TFAE

(i) φ is nonsingular

(ii) For each basis $\{v_i\}_{i=1}^n$ for V,

$\{\varphi(v_i)\}_{i=1}^n$ is a basis for W

(iii) $\exists$ basis $\{v_i\}_{i=1}^n$ of V s.t

$\{\varphi(v_i)\}_{i=1}^n$ is a basis for W.

pf (i) $\to$ (ii) Show $\{\varphi(v_i)\}_{i=1}^n$ spans W

For $w \in W$ $\exists$ $v \in V$ st $\varphi(v) = w$

Write $v = \sum_{i=1}^n \alpha_i v_i$ $\alpha_i \in F$

$w = \varphi(v) = \sum_{i=1}^n \alpha_i \varphi(v_i)$

Show $\{\varphi(v_i)\}_{i=1}^n$ lin indep. Given

$\sum \alpha_i \varphi(v_i) = 0$

$\varphi(\sum \alpha_i v_i)$

$\ker(\varphi) = 0$ so

$$\sum \alpha_i v_i = 0$$

$\{v_i\}_{i=1}^n$ is basis so

$$\alpha_i = 0 \qquad 1 \le i \le n.$$

(ii) $\to$ (iii) ✓

(iii) $\to$ (i) Routine. $\square$

10.2 Linear transformations and their matrix reps

For integers $m, n \geq 0$ define

$Mat_{m \times n}(F)$ = set of $m \times n$ matrices with entries in F

$Mat_{m \times n}(F)$ is a vector space over F.

this vector space has a basis

$$E_{ij} \qquad 1 \leq i \leq m, \qquad 1 \leq j \leq n$$

where

$$E_{ij} = \begin{array}{cc} \begin{array}{ccc} \quad & j & \quad \end{array} & \\ \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} & i \end{array}$$

The vector space $Mat_{m \times n}(F)$ has dimension mn.

Given fd vector spaces V, W over F

Recall

$$\mathrm{Hom}_F(V,W) = \text{set of all } F\text{-module homs } V \to W$$

[ie linear trans $V \to W$]

$\mathrm{Hom}_F(V,W)$ is an F-module

[ie vector space over F]

Given

basis $\{v_i\}_{i=1}^{n}$ for V $\quad *$

basis $\{w_i\}_{i=1}^{m}$ for W $\quad **$

Given

$$\varphi \in \mathrm{Hom}_F(V,W)$$

$\exists$ unique $A \in \mathrm{Mat}_{m \times n}(F)$ such that

$$\varphi(v_j) = \sum_{i=1}^{m} \underset{\substack{\uparrow \\ (i,j)\text{-entry of } A}}{A_{ij}} w_i \qquad 1 \le j \le n$$

Call A the <u>matrix that represents φ with respect to $*$ and $**$</u>

For $v \in V$ write

$$v = \sum_{i=1}^{n} \alpha_i v_i \qquad \alpha_i \in F$$

We have

$$\varphi(v) \in W$$

Write

$$\varphi(v) = \sum_{i=1}^{m} \beta_i w_i \qquad \beta_i \in F$$

LEM 1 With above notation,

$$A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

In other words

$$\sum_{j=1}^{n} A_{ij} \alpha_j = \beta_i \qquad 1 \le i \le m \qquad (\star)$$

pf We have

$$\sum_{i=1}^{m} \beta_i w_i = \varphi(v)$$

$$= \varphi\left(\sum_{j=1}^{n} \alpha_j v_j\right)$$

$$= \sum_{j=1}^{n} \alpha_j \varphi(v_j)$$

$$= \sum_{j=1}^{n} \alpha_j \left(\sum_{i=1}^{m} A_{ij} w_i\right)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} A_{ij} \alpha_j\right) w_i$$

For $1 \le i \le m$, compare w_i-coefs to get ☆ $\square$

Prop 2 With above notation, the map

$$\begin{aligned} \text{Hom}_F(V,W) &\longrightarrow \text{Mat}_{m\times n}(F) \\ \text{ч}: \quad \varphi &\longrightarrow \underbrace{\text{matrix rep } \varphi \text{ wrt } * \text{ and } **}_{\overset{\shortparallel}{\varphi^{\text{ч}}}} \end{aligned}$$

is an iso of vector spaces

pf First show ч is an F-module hom

show

$$(\varphi + \phi)^{\text{ч}} \overset{?}{=} \varphi^{\text{ч}} + \phi^{\text{ч}} \qquad \varphi, \phi \in \text{Hom}_F(V,W)$$

show $(\alpha\varphi)^{\text{ч}} \overset{?}{=} \alpha(\varphi^{\text{ч}}) \qquad \alpha \in F, \ \varphi \in \text{Hom}_F(V,W)$

Both are routine (ex)

Show ч is bijection.

Show ч is injective:

Given $\varphi \in \text{Hom}_F(V,W)$ st

$$\varphi^{\text{ч}} = 0$$

So the matrix $\varphi^{\mathcal{G}}$ has all entries 0

So $\quad \varphi(v_j) = 0 \qquad 1 \le j \le n$

So $\quad \varphi(v) = 0 \qquad v \in V$

So $\quad \varphi = 0$

Show $\mathcal{G}$ is surjective

Given $A \in Mat_{m \times n}(F)$

Define $\varphi \in Hom_F(V, W)$ st

$$\varphi(v_j) = \sum_{i=1}^{m} A_{ij}\, w_i \qquad 1 \le j \le n$$

By constr $\quad \varphi^{\mathcal{G}} = A.$

So $\mathcal{G}$ is surj.

$\square$