Lecture 23 Friday March 11

Ch 11 Vector Spaces

11.1 Basic theory

Fix a field F

A <u>vector space over F</u> is an F-module

Let V denote a vector space over F

The elements of V are called <u>vectors</u>

the elements of F are called <u>scalars</u>

DEF 1 Given a subset $S \subseteq V$,

(i) S is <u>linearly independent</u> whenever for all integers $n \geq 1$ and

$v_i \in S \quad \alpha_i \in F \quad 1 \leq i \leq n,$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$ implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$

(ii) S <u>spans</u> V whenever

$$V = FS$$

DEF 2 An <u>unordered basis</u> for V is a linearly independent set of vectors that span V.

An <u>ordered basis</u> for V is an unordered basis S for V, together with an ordering of S.

By a <u>basis</u> for V we mean an ordered basis for V.

Ex 3 Let x = indet. View the polynomial ring $F[x]$ as a vector space over F. Then the vectors

$$1, x, x^2, \ldots$$

form a basis for $F[x]$.

LEM 4 Given a finite subset $S \subseteq V$

Assume S spans V, but no proper subset of S spans V. Then S is an unordered basis for V,

pf Show S is lin indep.

Given scalars

$$\alpha_v \in F \qquad v \in S$$

st

$$0 = \sum_{v \in S} \alpha_v v \qquad *$$

Show $\alpha_v = 0 \quad \forall v \in S$

Suppose $\exists\, w \in S$ such that $\alpha_w \neq 0$

Solve $*$ for w:

$$w = \frac{-1}{\alpha_w} \sum_{v \in S \setminus w} \alpha_v v$$

Now

$S \setminus w$ spans V

Cont. $\square$

COR 5 Given a finite set A of vectors that span V, then A contains a subset that forms an unordered basis for V.

pf Consider the subsets S of A that span V. Of these, pick S with $|S|$ minimal. By constr S spans V but no proper subset of S spans V. Now S is a basis for V by LEM 4.

$\square$

Prop 6 Given an unordered basis A of V

Given lin indep vectors $b_1, b_2, \dots, b_m$ in V. Then

$\exists$ mutually dist vectors $a_1, a_2, \dots, a_m$ in A st each

of the following is a basis for V:

$$\{b_1, \dots, b_k\} \cup \overline{\{a_1, \dots, a_k\}} \qquad 0 \le k \le m$$

↑

$A \setminus \{a_1, \dots, a_k\}$

pf Induction on m

$m = 0$ ✓

$m \ge 1$: By ind $\exists$ mut dist $a_1, \dots, a_{m-1}$ in A st

$$\{b_1 \dots b_{m-1}\} \cup \overline{\{a_1, \dots, a_{m-1}\}}$$

is an unordered basis for V. Write

coefs unique

$$b_m = \sum_{i=1}^{m-1} \alpha_i b_i + \sum_{v \in \overline{\{a_1, \dots, a_{m-1}\}}} \alpha_v v \qquad (*)$$

So

coefs not all 0

$$\underbrace{b_m - \sum_{i=1}^{m-1} \alpha_i b_i}_{\neq 0} = \sum_{v \in \overline{\{a_1, \dots, a_{m-1}\}}} \alpha_v v$$

$\exists\ w \in \overline{\{a_1 \dots a_{m-1}\}}$ st $\alpha_w \neq 0$

Define

$$a_m = w$$

show

$$\{b_1, \dots, b_m\} \cup \overline{\{a_1, \dots, a_m\}} \qquad (**)$$

is unordered basis for V:

Solve $(*)$ for a_m

$$a_m \text{ is in the span of } \underbrace{\text{remaining vectors in } *}_{\substack{= \\ (**)}}$$

So $(**)$ spans V

show $(**)$ lin indep:

Given

$$\sum_{i=1}^{m} \beta_i b_i + \sum_{v \in \overline{\{a_1, \dots, a_m\}}} \beta_v v = 0 \qquad \beta_i, \beta_v \in F$$

Then $\beta_m = 0$, else uniqueness of coefs in $(*)$ is contradicted.

now

$$\{b_1, \dots, b_{m-1}\} \cup \overline{\{a_1, \dots, a_m\}} \qquad \text{lin dep}$$

$$\rotatebox{90}{$\subseteq$}$$

$$\{b_1, \dots, b_{m-1}\} \cup \overline{\{a_1, \dots, a_{m-1}\}} \qquad \text{lin dep}$$

Cont

So $(**)$ is lin indep and hence an unordered basis for V

$\square$

COR 7 Given a basis $a_1, a_2, \ldots, a_n$ of V then each lin indep subset of V contains at most n vectors.

pf Ref to Prop 6. obs $m \leq |A|$ $\square$

Prop 8 Given two bases for V:

$a_1, a_2, \dots, a_n$ $\qquad$ $b_1, b_2, \dots, b_m$

then $n = m$.

Pf By symmetry, WLOG $n \leq m$.
By Cor 7, $m \leq n$. So $n = m$. $\square$

DEF 9. We define

$$\text{dimension of } V = \begin{cases} n & \text{if } V \text{ has basis } a_1, \dots, a_n \\ \infty & \text{else} \end{cases}$$

LEM 10 Assume V has finite dim n.

Then each spanning set of V contains at least n vectors.

pf The spanning set contains an unordered basis for V by Cor 5. Result follows by Def 9. $\square$

Prop 11 Assume V has finite dim.

Given a lin indep subset $S \subseteq V$.

then S is contained in an unordered basis for V.

pf By assumption $\exists$ unordered basis A for V

Write $S = \{b_1, b_2, \ldots, b_m\}$

By Cor 7, $m \leq |A|$

By Prop 6, we can add $|A| - m$ vectors from A to S

to get an unordered basis for V. $\square$

For $n \in \mathbb{N}$ consider the free F-module

$$F^n = \underbrace{F \times F \times \cdots \times F}_{n}$$

F^n is vector space over F.
For $1 \le i \le n$ define

$$e_i = (0, \dots, 0, 1, 0, \dots, 0)$$
$$\uparrow_i$$

Obs $e_1, e_2, \dots, e_n$ is a basis for F^n.

LEM 12 Given a basis $a_1, a_2, \dots, a_n$ for V.

then the map

$$\varphi : \begin{array}{ccc} F^n & \longrightarrow & V \\ (\alpha_1, \dots, \alpha_n) & \longmapsto & \sum_{i=1}^{n} \alpha_i a_i \end{array}$$

is an iso of F-modules (ie vectorspace iso)

pf One checks

$$\varphi(u+v) = \varphi(u) + \varphi(v) \qquad u, v \in F^n.$$

$$\varphi(\alpha u) = \alpha \varphi(u) \qquad \alpha \in F \quad u \in F^n$$

so φ is F-module hom

φ is inj since $a_1 \dots a_n$ is lin indep.
φ is surj since .. spans V.
so φ is bijection. □