Lecture 22 Wednesday March 9

R denotes a ring with 1

LEM 7 Given R-module M

Given R-submodules A, B of M. TFAE

(i) $A + B = M$ and $A \cap B = 0$

(ii) the map

$$\psi : \begin{array}{ccc} A \times B & \to & M \\ (a, b) & \to & a+b \end{array}$$

is an R-module isomorphism.

pf (i) $\to$ (ii) ψ is R-module hom by LEM5

(with $\varphi =$ incl, $\phi =$ incl)

show ψ is bijection.

ψ is surj since $A+B = M$

show ψ is inj: Given $a \in A$ and $b \in B$ st

$$\underset{\substack{\| \\ a+b}}{\psi(a,b)} = 0$$

$$a = -b \in A \cap B = 0$$

So $a = 0 = b$

ψ is inj ✓

(ii) $\to$ (i) Routine

$\square$

LEM 8 Given R-module M

Given R-submodules of M:

$A_1, A_2, \ldots, A_t$ $\qquad t \geq 2$

Assume

$$M = A_1 + A_2 + \cdots + A_t \qquad (*)$$

TFAE

(i) For $a_i \in A_i$ $(1 \leq i \leq t)$,

$a_1 + a_2 + \cdots + a_t = 0$ implies $a_i = 0$ for $1 \leq i \leq t$

(ii) For $1 \leq i \leq t$, A_i has 0 intersection with

$$A_1 + \cdots + A_{i-1} + A_{i+1} + \cdots + A_t \qquad ☆$$

pf (i) $\to$ (ii) Given

$$x \in A_i \cap ☆$$

Show $x = 0$.

Write

$$x = a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_t \qquad a_j \in A_j$$

So $$0 = a_1 + \cdots + a_{i-1} + \underbrace{(-x)}_{\substack{\| \text{def} \\ a_i \in A_i}} + a_{i+1} + \cdots + a_t$$

So $$a_1 = a_2 = \cdots = a_t = 0$$

So $$x = -a_i = 0$$

(ii) $\to$ (i) Given $a_j \in A_j$ $(1 \le j \le t)$ st

$$a_1 + a_2 + \cdots + a_t = 0$$

For $1 \le i \le t$,

$$-a_i = a_1 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_t$$

$$\in A_i \cap \star$$

$$= 0$$

So $a_i = 0$ for $1 \le i \le t$. $\square$

DEF 9 Ref to LEM 8,

the sum $*$ is called <u>direct</u> whenever the conditions (i), (ii) hold. In this case write

$$M = A_1 \oplus A_2 \oplus \cdots \oplus A_t$$

Prop 10 Given an R-module M

Given R-submodules of M:

$A_1, A_2, \ldots, A_t$ $\qquad t \geq 2$

TFAE

(i) $M = A_1 \oplus A_2 \oplus \cdots \oplus A_t$

(ii) the map

$$A_1 \times A_2 \times \cdots \times A_t \longrightarrow M$$

$$(a_1, a_2, \ldots, a_t) \longrightarrow a_1 + a_2 + \cdots + a_t$$

is an R-module isomorphism.

pf The case $t=2$ is just LEM 7. For general t the pf is similar. $\square$

DEF 11 Given an R module M

Given a subset $A \subseteq M$ (poss $|A| = \infty$)

Define RA = set of all finite sums

$r_1 a_1 + r_2 a_2 + \cdots + r_t a_t$ $t \in \mathbb{N}$, $r_i \in R$, $a_i \in A$

Interp $RA = 0$ if $A = \phi$.

Obs RA is an R-submodule of M, said to be <u>generated by A</u>

Note Ref to Def 11, assume

$$|A| < \infty$$
$$\overset{\shortparallel}{t}$$

Write $A = \{a_1, a_2, \ldots, a_t\}$

then $RA = Ra_1 + Ra_2 + \cdots + Ra_t$

DEF 12 An R module M is called <u>finitely generated</u> whenever $\exists$ finite subset $A \subseteq M$ st $M = RA$.

Free R-modules

Recall R-module R

For $t \geq 1$ consider R-module

$$\underbrace{R \times R \times \cdots \times R}_{t} \qquad (*)$$

R-module $(*)$ called <u>free of rank t</u>

LEM 13 Given an R-module M

Given $a_1, a_2, \ldots, a_t \in M$ $\qquad t \geq 1$

Then the map

$$\begin{array}{ccc} R \times R \times \cdots \times R & \longrightarrow & M \\ (r_1, r_2, \ldots, r_t) & \longrightarrow & r_1 a_1 + r_2 a_2 + \cdots + r_t a_t \end{array} \qquad *$$

is an R-module homomorphism.

pf For $1 \leq i \leq t$ the map

$$\begin{array}{ccc} R & \longrightarrow & M \\ r & \longrightarrow & r a_i \end{array}$$

is an R-module hom.

The map $*$ is the composition

R-mod hom

$$\begin{array}{ccccc} R \times R \times \cdots \times R & \longrightarrow & M \times M \times \cdots \times M & \longrightarrow & M \\ (r_1, r_2, \ldots, r_t) & \longrightarrow & (r_1 a_1, r_2 a_2, \ldots, r_t a_t) & \longrightarrow & r_1 a_1 + r_2 a_2 + \cdots + r_t a_t \end{array}$$

$\square$

We now take a more abstract view of free R modules.

Given a set A (possibly $|A| = \infty$)

Define a set of functions $F(A)$ by

$$F(A) = \{ f : A \to R \mid f(a) \neq 0 \text{ for finitely many } a \in A \}$$

Define

$$\mathbb{O} : \begin{array}{l} A \to R \\ a \to 0 \end{array}$$

So

$$\mathbb{O} \in F(A)$$

For $f, g \in F(A)$ define

$$f+g : \begin{array}{l} A \to R \\ a \to f(a) + g(a) \end{array}$$

Obs

$$f+g \in F(A)$$

One checks

$$F(A),\ +,\ \mathbb{O}$$

is an abelian group.

For $r \in R$ and $f \in F(A)$ define

a function

$$rf : \begin{array}{l} A \to R \\ a \to rf(a) \end{array}$$

obs $\quad rf \in F(A)$

One checks $F(A)$ becomes an R-module with action

$$\begin{array}{ccc} R \times F(A) & \to & F(A) \\ r \quad f & \to & rf \end{array}$$

" free R-module on the set A "

Now assume $\underset{\overset{\shortparallel}{t}}{|A|} < \infty$

Compare $F(A)$ with R-module

$$\underbrace{R \times R \times \cdots \times R}_{t}$$

WLOG $A = \{1, 2, \ldots, t\}$

the map

$$\begin{array}{ccc} F(A) & \to & R \times R \times \cdots \times R \\ f & \to & \big(f(1), f(2), \ldots, f(t)\big) \end{array}$$

is an iso of R-modules (ex)

LEM 13 looks as follows from the abstract viewpt.

Prop 14 Given a set A

Consider the free R-module $F(A)$

Given any R-module M and any function

$$\varphi : A \to M$$

then the map

$$\hat{\varphi} : \begin{array}{ccc} F(A) & \longrightarrow & M \\ f & \longrightarrow & \sum_{a \in A} f(a)\varphi(a) \end{array}$$

is an R-module homomorphism.

pf For $f, g \in F(A)$ check

$$\hat{\varphi}(f+g) \overset{?}{=} \hat{\varphi}(f) + \hat{\varphi}(g)$$

$$\hat{\varphi}(f+g) = \sum_{a \in A} \underbrace{(f+g)(a)}_{= f(a)+g(a)} \varphi(a)$$

$$\underbrace{\hat{\varphi}(f) + \hat{\varphi}(g) = \sum_{a \in A} f(a)\varphi(a) + \sum_{a \in A} g(a)\varphi(a)}_{= \sum_{a \in A} (f(a)+g(a))\varphi(a)}$$

ok

For $r \in R$ and $f \in F(A)$ check

$$\hat{\varphi}(rf) \overset{?}{=} r\,\underbrace{\hat{\varphi}(f)}_{=\ \sum_{a \in A} f(a)\,\varphi(a)}$$

$$\hat{\varphi}(rf) = \underbrace{\sum_{a \in A} \underbrace{(rf)(a)}_{=\ r f(a)}\, \varphi(a)}_{=\ r \sum_{a \in A} f(a)\,\varphi(a)}$$

ok

$\square$