Lecture 21 Monday March 7

Let R = any ring (not nec com)

Given R-modules M, N

Given R-module homomorphism $\varphi : M \to N$

Let $K = \ker(\varphi)$

So K is R-submodule of M

and
$$\text{quot} : \begin{array}{ccc} M & \to & M/K \\ a & \to & a + K \end{array}$$

is R-module homo.

$$\begin{tikzcd} & M \arrow[dl, "\text{quot}"'] \arrow[dr, "\varphi"] & \\ M/K \arrow[rr, dashed, "?"'] & & N \end{tikzcd}$$

LEM 12 With above notation,

(i) $\exists$ unique R-module hom $\hat{\varphi} : M/K \to N$ s.t

$$\varphi : M \xrightarrow[\text{quot}]{} M/K \xrightarrow[\hat{\varphi}]{} N$$

(ii) $\hat{\varphi}$ is injective.

pf (i) For $a \in M$ require that

φ sends

$$a \longrightarrow a+K \longrightarrow \hat{\varphi}(a+K)$$

So

$$\varphi(a) = \hat{\varphi}(a+K)$$

So $\hat{\varphi}$ is unique if it exists.

Show $\hat{\varphi}$ exists.

Define a function

$$\hat{\varphi} : M/K \longrightarrow N$$
$$a+K \longrightarrow \varphi(a)$$

$\hat{\varphi}$ is well defined since $\varphi(K) = 0$

Show $\hat{\varphi}$ is R-module hom.

For $a, b \in M$

$$\hat{\varphi}(a+b+K) \overset{?}{=} \hat{\varphi}(a+K) + \hat{\varphi}(b+K)$$

$$\hat{\varphi}(a+K) = \varphi(a), \quad \hat{\varphi}(b+K) = \varphi(b)$$

$$\hat{\varphi}(a+b+K) = \varphi(a+b) = \varphi(a) + \varphi(b)$$

ok

For $r \in R$ and $a \in M$

$$r \underbrace{\hat{\varphi}(a+K)}_{\varphi(a)} = \underbrace{\hat{\varphi}(\underbrace{r(a+K)}_{ra+K})}_{\varphi(ra)}$$

$$\varphi(ra) = r\varphi(a)$$

ok

(ii) Given $a \in M$ st
$$\hat{\varphi}(a+K) = 0$$

We have
$$0 = \hat{\varphi}(a+K)$$
$$= \varphi(a)$$

So $a \in K$

Now $a+K = K$ = zero in M/K $\square$

10.3 Generation of modules, direct sums, free modules

In this section R is a ring with 1.

<u>Comment</u> Given an R-module M and an R-submodule A,

the inclusion map

$$\begin{array}{rccc} \text{incl} & A & \to & M \\ & a & \to & a \end{array}$$

is an R-module homomorphism (ex)

— o —

Recall R is an R-module with action

$$\begin{array}{ccc} R \times R & \to & R \\ r \quad a & \to & ra \end{array}$$

LEM 1 Given an R-module M

Given $a \in M$.

Then the map

$$\varphi \quad \begin{array}{ccc} R & \longrightarrow & M \\ r & \longmapsto & ra \end{array}$$

is an R-module homomorphism

[the image is denoted Ra]

pf For $r, s \in R$

$$\begin{array}{ccccc} \varphi(r+s) & \overset{?}{=} & \varphi(r) & + & \varphi(s) \\ \| & & \| & & \| \\ (r+s)a & & ra & & sa \\ \| & & & & \\ ra+sa & & \text{ok} & & \end{array}$$

$$\begin{array}{ccc} \varphi(rs) & \overset{?}{=} & r\underbrace{\varphi(s)}_{sa} \\ \| & & \| \\ (rs)a & & r(sa) \\ \| & & \\ r(sa) & & \text{ok} \end{array}$$

$\square$

Given an R-module M.

Call M <u>Cyclic</u> whenever $\exists\, a \in M$

st $M = Ra$.

LEM 2 Given R-modules A, B. Then the direct product $A \times B$ becomes an R module with action

$$R \times A \times B \longrightarrow A \times B$$

$$r \quad (a,b) \longrightarrow (ra, rb)$$

pf For x and x' in $A \times B$ write $x = (a,b)$ $x' = (a', b')$

show $r(x+x') \overset{?}{=} rx + rx'$ $r \in R$

$$rx + rx' = \underbrace{(ra, rb) + (ra', rb')}_{(ra+ra',\ rb+rb')}$$

$$r(x+x') = r(a+a', b+b') = (r(a+a'), r(b+b')) = (ra+ra', rb+rb')$$

ok

show $(r+s)x \overset{?}{=} rx + sx$

$$rx + sx = \underbrace{(ra, rb) + (sa, sb)}_{(ra+sa,\ rb+sb)}$$

$$(r+s)x = ((r+s)a, (r+s)b) = (ra+sa, rb+sb)$$

ok

Show

$$(rs)\,x \overset{?}{=} \underbrace{r\,(s\,x)}_{= \; r\cdot(sa,\, sb)}$$

$$(rs)\,x = \big((rs)a,\ (rs)b\big)$$

$$\underbrace{r\cdot(sa,\, sb)}_{(r(sa),\ r(sb))}$$

$(rs)a = r(sa)$, $\quad (rs)b = r(sb)$ ok

$$1\,x \overset{?}{=} x = (a, b)$$

$$1\,x = (1a,\ 1b)$$

$1a = a$, $\quad 1b = b$ ok

$\square$

LEM 3 Given R-modules A, A', B, B'

Given R-module homomorphisms

$$\psi : A \to A', \qquad \phi : B \to B'$$

Then the map

$$\Psi : \begin{array}{ccc} A \times B & \longrightarrow & A' \times B' \\ (a,b) & \longrightarrow & (\psi(a), \phi(b)) \end{array}$$

is an R-module hom.

Pf For x and y in $A \times B$

write $x = (a,b)$ $\qquad y = (\alpha, \beta)$

check

$$\Psi(x+y) \overset{?}{=} \underbrace{\Psi(x) + \Psi(y)}_{}$$

$$\Psi(x) = (\psi(a), \phi(b)), \qquad \Psi(y) = (\psi(\alpha), \phi(\beta))$$

$$\Psi(x+y) = \Psi(a+\alpha, b+\beta) = (\psi(a+\alpha), \phi(b+\beta)) = (\psi(a)+\psi(\alpha), \phi(\alpha)+\phi(\beta))$$

$$\Psi(x)+\Psi(y) = (\psi(a)+\psi(\alpha), \phi(b)+\phi(\beta))$$

ok

$$\Psi(rx) \overset{?}{=} r\,\underbrace{\Psi(x)}_{= (\psi(a), \phi(b))} \qquad r \in R$$

$$\Psi(rx) = \Psi(ra, rb) = (\psi(ra), \phi(rb)) = (r\psi(a), r\phi(b))$$

$$r\Psi(x) = (r\psi(a), r\phi(b))$$

ok

$\square$

LEM 4 For an R-module M

The map

$$\varphi : \begin{array}{ccc} M \times M & \longrightarrow & M \\ (a, b) & \longrightarrow & a+b \end{array}$$

is an R-module homomorphism

pf Given x and y in $M \times M$

write $x = (a, b)$ $\quad y = (\alpha, \beta)$

check

$$\varphi(x+y) \stackrel{?}{=} \varphi(x) + \varphi(y)$$

$$\varphi(x) = a+b \qquad \varphi(y) = \alpha+\beta$$

$$\varphi(x+y) = \varphi(a+\alpha, b+\beta) = a+\alpha+b+\beta$$

ok

$$\varphi(rx) \stackrel{?}{=} r\underbrace{\varphi(x)}_{a+b} \qquad r \in R$$

$$\varphi(rx) = \varphi(ra, rb) = ra + rb$$

$$r\varphi(x) = r(a+b) = ra + rb$$

ok

$\square$

LEM 5 Given R-modules A, B, M

Given R-module homomorphisms

$$\varphi : A \to M, \qquad \phi : B \to M$$

Then the map

$$\begin{aligned} \psi : \quad & A \times B \longrightarrow M \\ & (a, b) \longrightarrow \varphi(a) + \phi(b) \end{aligned}$$

is an R-module homomorphism.

pf ψ is the composition

R-module hom

$$\begin{array}{ccccc} A \times B & \longrightarrow & M \times M & \longrightarrow & M \\ (a, b) & \longrightarrow & (\varphi(a), \phi(b)) & \longrightarrow & \varphi(a) + \phi(b) \end{array}$$

$\square$

LEM 6 Given R-modules A, B

(i) Each of the maps

φ_A : $A \longrightarrow A \times B$, $a \to (a, 0)$ $\quad\quad$ φ_B : $B \longrightarrow A \times B$, $b \to (0, b)$

is an injective R-module hom.

(ii) Each of the maps

π_A : $A \times B \longrightarrow A$, $(a, b) \to a$ $\quad\quad$ π_B : $A \times B \longrightarrow B$, $(a, b) \to b$

is a surj R module hom.

pf (i) Consider $\varphi = \varphi_A$

For $a, \alpha \in A$

Check $\varphi(a+\alpha) \overset{?}{=} \varphi(a) + \varphi(\alpha)$

$$\begin{aligned}\varphi(a+\alpha) &= (a+\alpha, 0) = (a, 0) + (\alpha, 0) \\ \varphi(a) &= (a, 0), \quad \varphi(\alpha) = (\alpha, 0)\end{aligned}$$

OK

check

$$\varphi(ra) \overset{?}{=} r\varphi(a) \qquad r \in R$$

$$\varphi(ra) = (ra, 0)$$

$$r\varphi(a) = r(a, 0) = (ra, r0), \quad r0 = 0$$

ok

(ii) Consider $\pi = \pi_A$

For x and x' in $A \times B$

write $x = (a, b)$ $\quad$ $x' = (a', b')$

check

$$\pi(x + x') \overset{?}{=} \pi(x) + \pi(x')$$

$$\pi(x+x') = \pi(a + a', b + b') = a + a'$$

$$\pi(x) = a, \quad \pi(x') = a'$$

ok

check

$$\pi(rx) \overset{?}{=} r\pi(x) \qquad r \in R$$

$$\pi(rx) = \pi(r(a, b)) = \pi(ra, rb) = ra$$

$$r\pi(x) = ra$$

ok

$\square$