Lecture 20 Friday March 4

Given ring R (not nec commutative)

Given R-modules M, N

Recall

$$\mathrm{Hom}_R(M,N) = \{ \varphi \mid \varphi : M \to N \text{ is an } R\text{-module hom} \}$$

We saw:

- $\mathrm{Hom}_R(M,N)$, $+$, 0 is an abelian group
- For R commutative, $\mathrm{Hom}_R(M,N)$ is an R-module with action

$$\begin{aligned} R \times \mathrm{Hom}_R(M,N) &\to \mathrm{Hom}_R(M,N) \\ r \quad\; \varphi \quad &\to \quad r\varphi \end{aligned}$$

Next we consider $\mathrm{Hom}_R(M,N)$ as a ring

Given R-modules M, N, P

Given R-module homomorphisms

$\varphi : M \to N$ $\qquad$ $\phi : N \to P$

Obs that the composition

$$\phi \circ \varphi : \quad M \xrightarrow{\varphi} N \xrightarrow{\phi} P$$

is an R-module homomorphism.

Now assume $M = N = P$

For $\varphi, \phi \in Hom_R(M,M)$

we have $\phi \circ \varphi \in Hom_R(M,M)$

Define

$$End_R(M) = Hom_R(M,M)$$

"R-endomorphisms of M"

Define a function

$$\mathbb{1} : M \to M$$
$$a \to a$$

obs

$$\mathbb{1} \in End_R(M)$$

Note $\mathbb{1} \neq 0$ provided that $M \neq 0$

LEM 7 Given an R module M,

then $End_R(M)$ $\overset{\shortparallel}{\mathcal{R}}$

is a ring with composition product $\circ$ and identity $\mathbb{1}$

pf Recall $\mathcal{R}, +, 0$ is abel group

Composition product is assoc ✓

Distrib: For $\varphi, \phi, \theta \in \mathcal{R}$

$$(\varphi + \phi) \circ \theta \overset{?}{=} \varphi \circ \theta + \phi \circ \theta$$

For $a \in M$, show

$$((\varphi + \phi) \circ \theta)(a) \overset{?}{=} (\varphi \circ \theta + \phi \circ \theta)(a)$$

$$((\varphi + \phi) \circ \theta)(a) = (\varphi+\phi)(\theta(a)) = \varphi(\theta(a)) + \phi(\theta(a))$$

$$(\varphi \circ \theta + \phi \circ \theta)(a) = (\varphi \circ \theta)(a) + (\phi \circ \theta)(a) = \varphi(\theta(a)) + \phi(\theta(a))$$

OK

$$\theta \circ (\varphi + \phi) \overset{?}{=} \theta \circ \varphi + \theta \circ \phi$$

Fa $a \in M$

$$(\theta \circ (\varphi + \phi))(a) \overset{?}{=} (\theta \circ \varphi + \theta \circ \phi)(a)$$

$$(\theta \circ (\varphi + \phi))(a) = \theta((\varphi + \phi)(a)) = \theta(\varphi(a) + \phi(a)) = \theta(\varphi(a)) + \theta(\phi(a))$$

$$(\theta \circ \varphi + \theta \circ \phi)(a) = (\theta \circ \varphi)(a) + (\theta \circ \phi)(a) = \theta(\varphi(a)) + \theta(\phi(a))$$

ok

Fa $\varphi \in \mathcal{R}$ check

$$\varphi \circ \mathbb{1} = \mathbb{1} \circ \varphi = \varphi$$

Fa $a \in M$

$$(\varphi \circ \mathbb{1})(a) \overset{?}{=} \varphi(a)$$

$$(\varphi \circ \mathbb{1})(a) = \varphi(\mathbb{1}(a)) = \varphi(a)$$

✓

$$(\mathbb{1} \circ \varphi)(a) \overset{?}{=} \varphi(a)$$

$$\begin{aligned} &\parallel \\ &\mathbb{1}(\varphi(a)) \\ &\parallel \\ &\varphi(a) \end{aligned}$$

ok

$\square$

— o —

Assume R is commutative

Given R-module M

So $End_R(M)$ is a ring and R-module

Consider the function

$$\begin{aligned} \eta : \quad R &\longrightarrow End_R(M) \\ r &\longrightarrow r\mathbb{1} \end{aligned}$$

LEM 8 With the above notation,

(i) η is a homomorphism of rings

(ii) the image $\eta(R)$ is contained in the center of $\mathrm{End}_R(M)$

(iii) Assume R has a 1. Then

$$\eta(1) = \mathbb{1}$$

pf (i) For $r, s \in R$ check

$$\begin{array}{ccccc} \eta(r+s) & \overset{?}{=} & \eta(r) & + & \eta(s) \\ \| & & \| & & \| \\ & & r\mathbb{1} & & s\mathbb{1} \\ (r+s)\mathbb{1} & & & & \\ \| & & \text{ok} & & \\ r\mathbb{1} + s\mathbb{1} & & & & \end{array}$$

$$\begin{array}{ccc} z(rs) & \overset{?}{=} & z(r) \circ z(s) \\ \parallel & & \parallel \\ (rs)\mathbb{1} & & r\mathbb{1} \circ s\mathbb{1} \end{array}$$

Fa $a \in M$,

$$\begin{array}{ccc} ((rs)\mathbb{1})(a) & \overset{?}{=} & (r\mathbb{1} \circ s\mathbb{1})(a) \\ \parallel & & \parallel \\ (rs)\,\mathbb{1}(a) & & \underbrace{(r\mathbb{1})(\underbrace{(s\mathbb{1})(a)}_{\substack{\parallel \\ s(\mathbb{1}(a)) \\ \parallel \\ sa}})} \\ \parallel & & \\ (rs)a & & r(\mathbb{1}(sa)) \\ & & \parallel \\ & & r(sa) \\ & & \parallel \\ & & (rs)a \end{array}$$

ok

(ii) For $r \in R$ and $\varphi \in \mathrm{End}_R(M)$ show

$$z(r) \circ \varphi \stackrel{?}{=} \varphi \circ z(r)$$

For $a \in M$,

$$(z(r) \circ \varphi)(a) \stackrel{?}{=} (\varphi \circ z(r))(a)$$

$$\begin{aligned} (z(r) \circ \varphi)(a) &= z(r)(\varphi(a)) = r\mathbb{1}(\varphi(a)) = r\varphi(a) \\ (\varphi \circ z(r))(a) &= \varphi(z(r)(a)) = \varphi((r\mathbb{1})(a)) = \varphi(ra) = r\varphi(a) \end{aligned}$$

ok

(iii) $z(1) = 1\mathbb{1} = \mathbb{1}$

Since $\mathrm{End}_R(M)$ is an R-module.

$\square$

DEF 9 Assume R is commutative with 1

An <u>R-algebra</u> is a ring A together with a ring homomorphism

$$\eta : R \to A$$

such that

(i) $\eta(R)$ is contained in the center of A

(ii) $\eta(1) = 1$

COR 10 Assume R is commutative with 1.

Given R-module M.

then the ring $\mathrm{End}_R(M)$ from LEM 7 and the map η from LEM 8 form an R-algebra.

pf By LEM 8 and DEF 9. $\square$

For any ring R

Given R-module M

Given R-submodule N

For the abel groups M, N we have the quotient group M/N.

The quotient map

$$\varphi : \begin{array}{ccc} M & \longrightarrow & M/N \\ a & \longrightarrow & a+N \end{array}$$

is a surjective group homomorphism with kernel N

Next goal: turn M/N into an R-module st φ is an R-module homomorphism.

The R-module M/N must satisfy:

For $r \in R$ and $a \in M$

$$\underset{\substack{\| \\ ra+N}}{\varphi(ra)} = r \underbrace{\varphi(a)}_{\substack{\| \\ a+N}}$$

LEM 11 With above notation,

(i) M/N is an R-module with action

$$R \times M/N \to M/N$$
$$r \quad a+N \to ra+N$$

(ii) the quotient map $\varphi: M \to M/N$ is an R-module hom.

pf (i) Check axioms for R-module

Given $r, s \in R$ and $a \in M$

$$(r+s)(a+N) \overset{?}{=} \underbrace{\underbrace{r(a+N)}_{= ra+N} + \underbrace{s(a+N)}_{= sa+N}}_{ra+sa+N}$$

$$= (r+s)a + N$$
$$= ra+sa$$

ok

$$(rs)(a+N) \overset{?}{=} \underbrace{r(\underbrace{s(a+N)}_{= sa+N})}_{r(sa)+N}$$

$$= (rs)a + N \qquad r(sa) = (rs)a$$

ok

Fa $r \in R$ and $a, b \in M$

$$r(\underbrace{a+N + b+N}_{a+b+N}) \overset{?}{=} r(a+N) + r(b+N)$$

$$r(a+b)+N = ra+rb$$

$$\underbrace{ra+N}_{} \quad \underbrace{rb+N}_{}$$

$$r(a+N) + r(b+N) = ra+N + rb+N = ra+rb+N$$

ok.

(ii) By discussion above the lemma $\square$