Lecture 2 Fri Jan 22

Recall the commutative ring R and special subset $\mathcal{D} \subseteq R$. We constructed a ring Q with identity $\mathbb{1}$, and an injective ring homomorphism

$$i: \begin{array}{ccc} R & \longrightarrow & Q \\ a & \longrightarrow & \frac{ab}{b} \end{array} \qquad b \in \mathcal{D}$$

LEM 10 For $b \in \mathcal{D}$,

$i(b)$ is a unit in Q.

has a multiplicative inverse

Moreover

$$(i(b))^{-1} = \frac{d}{bd} \qquad d \in \mathcal{D}$$

pf
$$i(b)\,\frac{d}{bd} = \frac{bd}{d}\,\frac{d}{bd}$$
$$= \frac{bd^2}{bd^2}$$
$$= \mathbb{1}$$

□

LEM 11 For $\frac{a}{b} \in \mathbb{Q}$,

$$\frac{a}{b} = i(a)\, i(b)^{-1}$$

pf Fa $d \in \mathcal{D}$ view

$$i(a) = \frac{ad}{d} \qquad i(b) = \frac{bd}{d}$$

So

$$i(a)\, i(b)^{-1} = \frac{ad}{d} \; \frac{d}{bd}$$

$$= \frac{ad^2}{bd^2}$$

$$= \frac{a}{b} \qquad \square$$

LEM 12 Let S denote a commutative ring with $1 \neq 0$.

Let $\varphi : R \to S$ denote an injective ring hom such that $\varphi(d)$ is a unit in S for all $d \in \mathcal{D}$.

Then:

(i) $\exists$ ring hom $\phi : \mathcal{Q} \to S$ that sends

$$\frac{a}{b} \longrightarrow \varphi(a)\,\varphi(b)^{-1} \qquad \frac{a}{b} \in \mathcal{Q}$$

(ii) ϕ is injective

(iii) ϕ is the unique ring hom that makes this diagram commute:

$$\begin{tikzcd} R \arrow[dr, "\varphi"] \arrow[d, "i"'] & \\ \mathcal{Q} \arrow[r, "\phi"'] & S \end{tikzcd} \qquad *$$

Pf (i) For $\frac{a}{b}, \frac{A}{B} \in Q$ s.t.

$$\frac{a}{b} = \frac{A}{B}$$

check

$$\varphi(a)\ \varphi(b)^{-1} = \varphi(A)\ \varphi(B)^{-1}$$

We have

$$aB = Ab$$

Apply φ

$$\begin{array}{ccc} \varphi(aB) & = & \varphi(Ab) \\ \| & & \| \\ \varphi(a)\ \varphi(B) & & \varphi(A)\ \varphi(b) \end{array}$$

ok

$\exists$ a map

$$\phi : \begin{array}{ccc} Q & \longrightarrow & S \\ \frac{a}{b} & \longrightarrow & \varphi(a)\ \varphi(b)^{-1} \end{array}$$

Show ϕ is ring hom.

$$\phi(0) = \phi\left(\frac{0}{b}\right) = \varphi(0)\,\varphi(b)^{-1} = 0\,\varphi(b)^{-1} = 0$$

$b \in \mathcal{D}$

For $x, y \in \mathbb{Q}$,

$$\phi(x+y) \overset{?}{=} \phi(x) + \phi(y) \qquad \left[\text{ Write } x = \frac{a}{b},\ y = \frac{c}{d} \right]$$

$$\phi(x+y) = \phi\left(\frac{ad+bc}{bd}\right)$$

$$= \varphi(ad+bc)\,\varphi(bd)^{-1}$$

$$= \big(\varphi(a)\varphi(d) + \varphi(b)\varphi(c)\big)\,\varphi(b)^{-1}\varphi(d)^{-1}$$

$$= \varphi(a)\varphi(b)^{-1} + \varphi(c)\varphi(d)^{-1}$$

$$\phi(x) + \phi(y) = \varphi(a)\varphi(b)^{-1} + \varphi(c)\varphi(d)^{-1}$$

// ok

$$\phi(\mathbb{1}) = \phi\left(\frac{b}{b}\right) = \varphi(b)\,\varphi(b)^{-1} = 1$$

$b \in \mathcal{D}$

For $x, y \in \mathbb{Q}$,

$$\phi(xy) \overset{?}{=} \phi(x)\,\phi(y)$$

$\|$ $\quad\|$ [write $x = \frac{a}{b}$, $y = \frac{c}{d}$]

$$\phi\left(\frac{a}{b}\,\frac{c}{d}\right) \qquad \varphi(a)\,\varphi(b)^{-1}\,\varphi(c)\,\varphi(d)^{-1}$$

$\|$

$$\phi\left(\frac{ac}{bd}\right)$$

$\|$ ok

$\|$

$$\varphi(ac)\,\varphi(bd)^{-1}$$

$\|$

$$\varphi(a)\,\varphi(c)\,\varphi(b)^{-1}\,\varphi(d)^{-1}$$

(ii) Given $x \in \mathbb{Q}$ such that $\phi(x) = 0$.

Show $x = \textcircled{0}$

Write $x = \frac{a}{b}$

We have

$$0 = \phi\left(\frac{a}{b}\right)$$
$$= \varphi(a)\,\varphi(b)^{-1}$$

so
$$\varphi(a) = 0\,\varphi(b)$$
$$= 0$$

But φ is injective so

$$a = 0$$

Now
$$x = \frac{0}{b}$$
$$= \textcircled{0}$$

✓

(iii) Show $*$ commutes

For $a \in R$ chase a around diagram

$$\begin{tikzcd} a \arrow[d, "i"'] \arrow[r] & \varphi(a) \arrow[d, phantom, "\overset{?}{=}"] \\ \frac{ab}{b} \arrow[r] & \varphi(ab)\,\varphi(b)^{-1} \end{tikzcd} \qquad b \in \mathcal{D}$$

$$\varphi(ab)\,\varphi(b)^{-1} = \varphi(a)\,\varphi(b)\,\varphi(b)^{-1} = \varphi(a) \checkmark$$

Show ϕ is unique:

Let $\bar{\Phi} : Q \to S$ denote any ring hom that makes $*$ commute. Show $\bar{\Phi} = \phi$

Fn $x \in Q$ show

$$\bar{\Phi}(x) = \phi(x)$$

Write $x = \frac{a}{b}$

Since Φ makes $*$ commute,

$$\Phi\left(\frac{ab}{b}\right) = \varphi(a) \qquad a \in R, \quad b \in \mathcal{D}$$

For $x \in Q$ show

$$\Phi(x) = \phi(x)$$

Write $x = \frac{a}{b}$

$$\Phi\left(\frac{a}{b}\right) \stackrel{?}{=} \phi\left(\frac{a}{b}\right)$$

$$= \varphi(a)\,\varphi(b)^{-1}$$

Require

$$\Phi\left(\frac{a}{b}\right) \underbrace{\varphi(b)}_{\phi\left(\frac{bd}{d}\right)} \stackrel{?}{=} \varphi(a)$$

$$d \in \mathcal{D}$$

$$\underbrace{\Phi\left(\frac{a}{b}\right)\phi\left(\frac{bd}{d}\right)}$$

$$= \Phi\left(\frac{a}{b}\,\frac{bd}{d}\right)$$

$$= \Phi\left(\frac{abd}{bd}\right)$$

ok

$$= \varphi(a)$$

$\square$

We often identify R with a subring of Q via the injection i.

From this pt of view each element of $\mathcal{D}$ is a unit in Q, and

$$Q = \{ ab^{-1} / \ a \in R, \ b \in \mathcal{D} \}$$

Call Q the <u>ring of fractions</u> for $R, \mathcal{D}$

We sometimes write

$$Q = \mathcal{D}^{-1} R$$

Special Case

Recall an integral domain is a commutative ring with $1 \neq 0$ that has no 0-divisors.

Assume R is an integral domain.

Define $\mathcal{D} = R \setminus \{0\}$

Embed R in $Q = \mathcal{D}^{-1}R$ as above

In Q, each non0 element of R is a unit, and

$$Q = \{ ab^{-1} \mid a, b \in R, \quad b \neq 0 \}$$

So each non0 element of Q is a unit. Thus

Q is a field

" Field of fractions for R "

Given a field K

Given a subset $A \subseteq K$

Define

$F =$ intersection of all the subfields of K that contain A

Obs

F is a subfield of K that contains A

Call F the <u>subfield of K generated by A</u>

Prop 13 Assume R is an integral domain

Given a field K that contains R as a subring.

Then the following fields are isomorphic:

(i) the field of fractions $\mathcal{Q}$ of R,

(ii) the subfield F of K generated by R.

pf Obs

$$F = \{ ab^{-1} \mid a, b \in R, \ b \neq 0 \}$$

In LEM 12 take $S = K$ with $\varphi : R \to K$ the inclusion map. By LEM 12 $\exists$ injective ring hom

$\phi : \mathcal{Q} \to K$ that sends

$$\frac{a}{b} \to ab^{-1} \qquad a, b \in R \quad b \neq 0$$

So

$$F = \text{image of } \mathcal{Q} \text{ under } \phi.$$

Now the map

$$\phi : \mathcal{Q} \to F$$

is a field isomorphism

$\square$