3/2/16

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Lecture 19 Wednesday March 2

Given a ring R (not nec commutative)

Given R-module M

<u>Comments about M</u>

- For $r \in R$

$$r0 = r(0+0) = r0 + r0$$

So $r0 = 0$

- Also for $a \in M$

$$a + -a = 0$$

$$r(a + -a) = r0 = 0$$
$$\shortparallel$$
$$ra + r(-a)$$

So $r(-a) = -ra$

- Given R-submodules A, B of M

Define
$$A+B = \{ a+b \mid a \in A,\ b \in B \}$$

Note that $A+B$ is an R-submodule of M.

R- Module homomorphisms

Motivation

Given abel groups M, N

Given group homomorphism $\varphi : M \to N$

Recall $\varphi(a+b) = \varphi(a) + \varphi(b)$ $\quad a, b \in M$

View M, N as $\mathbb{Z}$-modules

$$\varphi(2a) = \varphi(a+a) \qquad a \in M$$
$$= \varphi(a) + \varphi(a)$$
$$= 2\varphi(a)$$

More generally $\quad n \in \mathbb{Z} \quad a \in M$

$$\varphi(na) = n\varphi(a)$$

Given field F

Given vector spaces V, W over F

Given a linear transformation $\varphi: V \to W$

Recall

$$\varphi(a+b) = \varphi(a) + \varphi(b) \qquad a, b \in V$$

$$\varphi(\alpha a) = \alpha \varphi(a) \qquad \alpha \in F \quad a \in V$$

Def 1 Given a ring R and R-modules M, N.

A function $\varphi: M \to N$ is an <u>R module homomorphism</u> whenever

$$\varphi(a+b) = \varphi(a) + \varphi(b) \qquad a, b \in M$$

$$\varphi(ra) = r\varphi(a) \qquad r \in R \quad a \in M$$

Ex Given R-modules M, N

The function

$$\Phi : M \to N$$
$$a \to 0$$

is an R-module homomorphism.

LEM 2 Given R-modules M, N

Given an R-module hom $\varphi : M \to N$

Define

$$K = \{ a \in M \mid \varphi(a) = 0 \}$$ "the kernel of φ"

Then K is an R-submodule of M

pf Check K is a subgroup of M

$$\varphi(0) = \varphi(0+0) = \varphi(0) + \varphi(0)$$

so $\varphi(0) = 0$

$0 \in K$

For $a, b \in K$ show $a+b \in K$:

$$\varphi(a+b) = \underset{\overset{\shortparallel}{0}}{\varphi(a)} + \underset{\overset{\shortparallel}{0}}{\varphi(b)} = 0$$

For $r \in R$ and $a \in K$ show

$ra \in K$:

$$\varphi(ra) = r \underset{\overset{\shortparallel}{0}}{\varphi(a)} = r0 = 0 \quad \checkmark$$

$\square$

LEM 3 Given R-modules M, N

Given R-module hom $\varphi: M \to N$

Define $\varphi(M) = \{ x \in N \mid \exists\, a \in M \text{ st } \varphi(a) = x \}$

"Image of φ"

then $\varphi(M)$ is an R-submodule of N.

pf (ex.) $\square$

DEF 4. Given R-modules M, N

A function $\varphi: M \to N$ is an <u>isomorphism of R-modules</u>

whenever f is a homomorphism of R-modules

and f is a bijection.

Given R-modules M, N. Define

$$\mathrm{Hom}_R(M,N) = \left\{ \varphi \;/\; \begin{array}{c}\varphi : M \to N \text{ is an} \\ R\text{-module hom}\end{array} \right\} \qquad \star$$

Next goal: investigate $\star$.

Is this a group? R-module? ring? or what?

Addition in $\mathrm{Hom}_R(M,N)$

Given $\varphi, \phi \in \mathrm{Hom}_R(M,N)$ define a function

$$\varphi + \phi : \begin{array}{l} M \longrightarrow N \\ a \to \varphi(a) + \phi(a) \end{array}$$

Check if $\varphi + \phi \in \mathrm{Hom}_R(M,N)$:

For $a, b \in M$

$$(\varphi + \phi)(a+b) \overset{?}{=} (\varphi + \phi)(a) + (\varphi + \phi)(b)$$

$(\varphi + \phi)(a) = \varphi(a) + \phi(a)$, $(\varphi + \phi)(b) = \varphi(b) + \phi(b)$

$$(\varphi + \phi)(a+b) = \varphi(a+b) + \phi(a+b)$$

$\varphi(a+b) = \varphi(a) + \varphi(b)$, $\phi(a+b) = \phi(a) + \phi(b)$ ok

For $r \in R$ and $a \in M$,

$$(\varphi + \phi)(ra) \overset{?}{=} r \underbrace{(\varphi + \phi)(a)}_{\varphi(a) + \phi(a)}$$

$$\begin{aligned} &\parallel \\ &\varphi(ra) + \phi(ra) \\ &\parallel \qquad\quad \parallel \\ &\underbrace{r\varphi(a) \qquad r\phi(a)} \\ &\parallel \\ &r(\varphi(a) + \phi(a)) \end{aligned}$$

ok

So

$$\varphi + \phi \in Hom_R(M, N)$$

Recall the map

$$0 \in Hom_R(M, N)$$

sends $a \to 0 \quad \forall a \in M$

LEM 5 Given R-modules M, N.

Then $\mathrm{Hom}_R(M, N)$, $+$, $\mathbb{O}$ is an abelian group.

pf By constr

$$\varphi + \mathbb{O} = \mathbb{O} + \varphi = \varphi \qquad \forall \varphi \in \mathrm{Hom}_R(M, N)$$

Also

$+$ is associative for $\mathrm{Hom}_R(M, N)$

since it is assoc for M, N.

For $\varphi \in \mathrm{Hom}_R(M, N)$ define a function

$$-\varphi : \begin{array}{l} M \to N \\ a \to -\varphi(a) \end{array}$$

Show $-\varphi \in \mathrm{Hom}_R(M, N)$

For $a, b \in M$,

$$(-\varphi)(a+b) \overset{?}{=} (-\varphi)(a) + (-\varphi)(b)$$

$$(-\varphi)(a) = -\varphi(a), \qquad (-\varphi)(b) = -\varphi(b)$$

$$(-\varphi)(a+b) = -\varphi(a+b) = -(\varphi(a) + \varphi(b)) = -\varphi(a) - \varphi(b)$$

ok

For $r \in R$ and $a \in M$ check

$$(-\varphi)(ra) \stackrel{?}{=} r\,\underbrace{(-\varphi)(a)}$$

$$\begin{array}{ccc} (-\varphi)(ra) & & r\,(-\varphi)(a) \\ \| & & \| \\ -\varphi(ra) & & -\varphi(a) \\ & & \| \\ -r\,\varphi(a) & \text{ok} & -r(\varphi(a)) \end{array}$$

So

$$-\varphi \in Hom_R(M, N)$$

By constr $\quad \varphi + (-\varphi) = (-\varphi) + \varphi = 0$

$\square$

Given R-modules M, N

Next consider $Hom_R(M,N)$ as an R-module

For $\varphi \in Hom_R(M,N)$ and $r \in R$

define a function

$$r\varphi : \begin{array}{l} M \to N \\ a \to r\,\varphi(a) \end{array}$$

check if $r\varphi \overset{?}{\in} Hom_R(M,N)$

For $a, b \in M$

$$(r\varphi)(a+b) \overset{?}{=} (r\varphi)(a) + (r\varphi)(b)$$

$$(r\varphi)(a) = r\varphi(a), \qquad (r\varphi)(b) = r\varphi(b)$$

$$(r\varphi)(a+b) = r(\varphi(a+b)) = r(\varphi(a) + \varphi(b)) = r\varphi(a) + r\varphi(b)$$

ok

For $s \in R$ and $a \in M$,

$$
\begin{array}{ccc}
(r\varphi)(sa) & \overset{?}{=} & s\,(r\varphi)\,(a) \\
\| & & \| \\
r(\varphi(sa)) & & s(r\,\varphi(a)) \\
& & \| \\
r(s\,\varphi(a)) & & (sr)\,\varphi(a) \\
\| & & \\
(rs)\,\varphi(a) & &
\end{array}
$$

Require $rs = sr$

So

$r\varphi \in Hom_R(M,N)$ if R is commutative.

LEM 6 Assume R is commutative.

Given R-modules M, N. Then

$$\mathrm{Hom}_R(M,N)$$

becomes an R-module with action

$$R \times \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M,N)$$
$$r \quad \varphi \longrightarrow r\varphi$$

pf For $r, s \in R$ and $\varphi \in \mathrm{Hom}_R(M,N)$

show

$$(r+s)\varphi \overset{?}{=} r\varphi + s\varphi$$

For $a \in M$

$$((r+s)\varphi)(a) \overset{?}{=} (r\varphi + s\varphi)(a)$$

$$((r+s)\varphi)(a) = (r+s)\varphi(a) = r\varphi(a) + s\varphi(a) \quad \text{ok}$$

$$(r\varphi + s\varphi)(a) = (r\varphi)(a) + (s\varphi)(a) = r\varphi(a) + s\varphi(a)$$

Next check

$$(rs)\varphi \stackrel{?}{=} r(s\varphi)$$

For $a \in M$ show

$$((rs)\varphi)(a) \stackrel{?}{=} (r(s\varphi))(a)$$

$$((rs)\varphi)(a) = (rs)\varphi(a)$$

$$(r(s\varphi))(a) = r(s\varphi)(a) = r(s\varphi(a)) = (rs)\varphi(a)$$

ok

For $r \in R$ and φ, ϕ in $Hom_R(M, N)$ show

$$r(\varphi + \phi) \stackrel{?}{=} r\varphi + r\phi$$

For $a \in M$

$$(r(\varphi+\phi))(a) \stackrel{?}{=} (r\varphi + r\phi)(a)$$

$$(r(\varphi+\phi))(a) = r((\varphi+\phi)(a)) = r(\varphi(a) + \phi(a)) = r\varphi(a) + r\phi(a)$$

$$(r\varphi + r\phi)(a) = (r\varphi)(a) + (r\phi)(a) = r\varphi(a) + r\phi(a)$$

ok

Now assume R has 1

Show

$$1\,\varphi \stackrel{?}{=} \varphi \qquad \varphi \in Hom_R(M,N)$$

For $a \in M$

$$(1\,\varphi)(a) \stackrel{?}{=} \varphi(a)$$

$\quad \parallel$

$1\,\varphi(a) \qquad$ ok

$\quad \parallel$

$\varphi(a)$

$\square$