Lecture 12 Monday Feb 15

Ch 9

Until further notice,

R is a commutative ring with $1 \neq 0$

Let $x =$ indeterminate

$R[x] =$ ring of polynomials in x with coefs in R

View R is a subring of $R[x]$

Fa $0 \neq f \in R[x]$ write

leading coef

$$f = a_0 + a_1 x + \cdots + \underset{\neq 0}{a_n} x^n \leftarrow \text{degree}$$

Declare $\deg(0) = 0$

LEM 1 TFAE

(i) R is an integral domain

(ii) $R[x]$ is an integral domain

pf (i) $\to$ (ii) Given nonzero $f, g \in R[x]$

show $fg \neq 0$

Write

$$f = a_0 + a_1 x + \cdots + \underset{\neq 0}{a_r} x^r \qquad a_i \in R$$

$$g = b_0 + b_1 x + \cdots + \underset{\neq 0}{b_s} x^s \qquad b_j \in R$$

$$fg = \left(\sum_{i=0}^{r} a_i x^i \right)\left(\sum_{j=0}^{s} b_j x^j \right)$$

$$= \sum_{i=0}^{r} \sum_{j=0}^{s} a_i b_j \, x^{i+j}$$

$$= a_r b_s \, x^{r+s} + \text{Lower terms}$$

$a_r b_s \neq 0$ since R is int domain.

so

$fg \neq 0$

(ii) $\to$ (i) Since R is subring of $R[x]$

$\square$

LEM 2 For an ideal I of R the following sets are equal:

(i) the ideal of $R[x]$ generated by I $= J_1$

(ii) the polynomials in x that have all coefs in I $= J_2$

[We call this common set $I[x]$]

pf $J_1 \subseteq J_2$:

Given $g \in J_1$

Write
$$g = f_1 a_1 + f_2 a_2 + \cdots + f_r a_r \qquad f_i \in R[x], \quad a_i \in I$$

Show each term is in J_2

For $f \in R[x]$ and $a \in I$

show $fa \in J_2$

write
$$f = b_0 + b_1 x + \cdots + b_s x^s \qquad b_i \in R \qquad s \geq 0$$

so
$$fa = \underbrace{b_0 a}_{\in I} + \underbrace{b_1 a}_{\in I} x + \cdots + \underbrace{b_s a}_{\in I} x^s \in J_2$$

$J_2 \subseteq J_1$: For $g \in J_2$ write
$$g = a_0 + a_1 x + \cdots + a_t x^t \qquad a_i \in I \qquad t \geq 0$$

$$g = \sum_{i=0}^{t} \underset{\in I}{a_i} \, \underset{\in R[x]}{x^i} \in J_1$$

$\square$

Given ring hom

$$\varphi : R \longrightarrow S$$

Then $\exists$ ring hom

$$\hat{\varphi} : R[x] \longrightarrow S[x]$$

$$\sum_i a_i x^i \longrightarrow \sum_i \varphi(a_i) x^i$$

- φ is surj iff $\hat{\varphi}$ is surj
- Let $I = \ker(\varphi)$. Then
  $\ker(\hat{\varphi}) = I[x]$

Given ideal I of R

$\swarrow$ $\qquad\qquad\qquad$ $\searrow$

ideal $I[x]$ of $R[x]$ $\qquad\qquad$ quot ring R/I

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad$ $\downarrow$

quot ring $R[x]/I[x]$ $\qquad\qquad$ poly ring $R/I\,[x]$

$\nwarrow$ $\nearrow$

compare.

LEM 3 The following rings are isomorphic:

$$R[x]/I[x], \qquad R/I\,[x]$$

pf Write $S = R/I$

Consider quotient map

$$\varphi \quad \begin{array}{ccc} R & \longrightarrow & S \\ r & \longmapsto & r+I \end{array}$$

φ is surj with kernel I.

Now

$$\hat{\varphi} : R[x] \to S[x]$$

is surj with ker $I[x]$

Now $\hat{\varphi}$ induces ring iso

$$R[x]/I[x] \longrightarrow S[x]$$

$$f + I[x] \longmapsto \hat{\varphi}(f)$$

Result follows. $\square$

LEM 4 Given an ideal I of R. Consider ideal $I[x]$ of $R[x]$. TFAE

(i) I is prime

(ii) $I[x]$ is prime.

pf I prime

$\Leftrightarrow$ R/I is integral domain by LEM 1

$\Leftrightarrow$ $R/I\,[x]$ is int. domain by LEM 3

$\Leftrightarrow$ $R[x]/I[x]$ is int domain.

$\Leftrightarrow$ $I[x]$ is prime. $\square$

Given mutually commuting indeterminates

$x_1, x_2, \ldots, x_n$

Define

$R[x_1, x_2, \ldots, x_n]$ = ring of polynomials in $x_1, x_2, \ldots x_n$ that have all coeffs in R

View

$$R[x_1, x_2, \ldots, x_n] = R[x_1, \ldots, x_{n-1}][x_n]$$

A monomial in $R[x_1, \ldots x_n]$ has form

$$x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n} \qquad e_i \geq 0 \quad (1 \leq i \leq n)$$

This monomial has

degree sequence $(e_1, e_2, \ldots, e_n)$

total degree $e_1 + e_2 + \cdots + e_n$
(or just degree)

Each $f \in R[x_1, \dots, x_n]$ is a finite sum

$$f = \sum_{i=1}^{\ell} a_i f_i \qquad \ell \geq 0$$

where

$0 \neq a_i \in R \quad (1 \leq i \leq \ell)$,

$f_1, f_2, \dots, f_\ell$ are mut dist monomials

Define

$$\deg(f) = \max\{\deg(f_i) \mid 1 \leq i \leq \ell\}$$

Call f homogenous of degree k whenever

$$\deg(f_i) = k \qquad 1 \leq i \leq \ell$$

- Each element of $R[x_1 \dots x_n]$ is a sum of homogenous elements.

Given a field F

We have seen $F[x]$ is Euclidean domain.

We now consider this in more detail.

LEM 5 Given field F

Given $a, b \in F[x]$ with $b \neq 0$

Then $\exists$ unique pair $q, r \in F[x]$ st

$$a = bq + r \qquad *$$

and

$$r = 0 \quad \text{or} \quad \deg(r) < \deg(b) \qquad **$$

pf Show q, r exist.

Call the ordered pair a, b a counterexample whenever q, r do not exist.

Assume a, b is a counterexample.

Obs $\deg(a) \geq \deg(b)$

else $*, **$ satisfied by

$$r = a; \qquad q = 0$$

Among all counterexamples, WLOG assume

$$\deg(a) - \deg(b) \text{ is minimal.}$$

Write

$$a = a_0 + a_1 x + \cdots + \underset{\substack{\#\\0}}{a_r} x^r \qquad a_i \in F$$

$$b = b_0 + b_1 x + \cdots + \underset{\substack{\#\\0}}{b_s} x^s \qquad b_j \in F$$

$$0 \le s \le r$$

$b_s \neq 0$, so b_s^{-1} exists in F

Define

$$A = a - b\, a_r b_s^{-1} x^{r-s} \quad \in F[x]$$

By constr: for A coef of x^r is 0
A does not involve $x^r, x^{r+1}, x^{r+2}, \ldots$

so

$$A = 0 \quad \text{or} \quad \deg(A) < r = \deg(a)$$

So A, b is not a counterexample

So $\exists\ Q, r \in F[x]$ st

$$A = bQ + r$$

and

$$r = 0 \quad \text{or} \quad \deg(r) < \deg(b)$$

We have

$$A = bQ + r$$

$$\parallel$$

$$a - b a_r b_s^{-1} x^{r-s}$$

$$a = b\Big(\underbrace{Q + a_r b_s^{-1} x^{r-s}}_{\substack{\parallel \\ \text{call this } q}}\Big) + r$$

Now q, r satisfy $*, **$.

So a, b not a counterexample. Cont.

We have shown q, r exists.

Next we show q, r is unique.

Suppose $\exists\ q', r' \in F[x]$ st

$$a = bq' + r' \qquad ☆$$

and

$$r' = 0 \quad \text{or} \quad \deg(r') < \deg(b) \qquad ☆☆$$

show $q = q'$ and $r = r'$. Assume $q \neq q'$ else done.

Combine $*$, ☆ to get

$$b(q - q') = r' - r$$

So

$$\deg(b) + \deg(q-q') = \deg(r'-r)$$

We have

$$\underbrace{\deg(b) + \overset{\geq 0}{\deg(q-q')}}_{\geq \deg(b)} = \deg(r'-r) < \deg(b)$$

by $**$, ☆☆

cnt.

So $q = q'$ and $r = r'$ $\square$