

Lecture 12 Monday Feb 15

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Ch 9

Until further notice,

R is a commutative ring with $1 \neq 0$

Let $x = \text{indeterminate}$

$R[x] = \text{ring of polynomials in } x \text{ with coeffs in } R$

View

R is a subring of $R[x]$

For $0 \neq f \in R[x]$ write

$$f = a_0 + a_1 x + \dots + \underbrace{a_n x^n}_{\substack{\text{leading coef} \\ \leftarrow \text{degree}}} \\ \# \\ 0$$

Declare

$$\deg(0) = 0$$

LEM 1 TFAE

(i) R is an integral domain(ii) $R[x]$ is an integral domain

pf (i) \rightarrow (ii) Given n, m $f, g \in R[x]$
 show $fg \neq 0$

Write

$$f = a_0 + a_1x + \dots + \underset{\neq 0}{a_r}x^r \quad a_i \in R$$

$$g = b_0 + b_1x + \dots + \underset{\neq 0}{b_s}x^s \quad b_j \in R$$

$$fg = \left(\sum_{i=0}^r a_i x^i \right) \left(\sum_{j=0}^s b_j x^j \right)$$

$$= \sum_{i=0}^r \sum_{j=0}^s a_i b_j x^{i+j}$$

$$= a_r b_s x^{r+s} + \text{Lower terms}$$

$a_r b_s \neq 0$ since R is int domain.

So

$$fg \neq 0$$

(ii) \rightarrow (i) Since R is subring of $R[x]$

□

LEM 2 For an ideal I of R the following sets

are equal:

$= J_1$

(i) the ideal of $R[x]$ generated by I

(ii) the polynomials in x that have all coeffs in I

$= J_2$

[We call this common set $I[x]$]

pf $J_1 \subseteq J_2$:

Given $g \in J_1$

write

$g = f_1 a_1 + f_2 a_2 + \dots + f_r a_r$

$f_i \in R[x], a_i \in I$

Show each term is in J_2

For $f \in R[x]$ and $a \in I$

show $fa \in J_2$

write

$f = b_0 + b_1 x + \dots + b_n x^n$

$b_i \in R, n \geq 0$

so

$fa = \underbrace{b_0 a}_I + \underbrace{b_1 a}_I x + \dots + \underbrace{b_n a}_I x^n \in J_2$

$J_2 \subseteq J_1$:

For $g \in J_2$ write

$g = a_0 + a_1 x + \dots + a_t x^t$

$a_i \in I, t \geq 0$

$g = \sum_{i=0}^t \underbrace{a_i}_I x^i \in J_1$

□

Given ring hom

$$\varphi: R \rightarrow S$$

Then \exists ring hom

$$\hat{\varphi}: R[x] \rightarrow S[x]$$

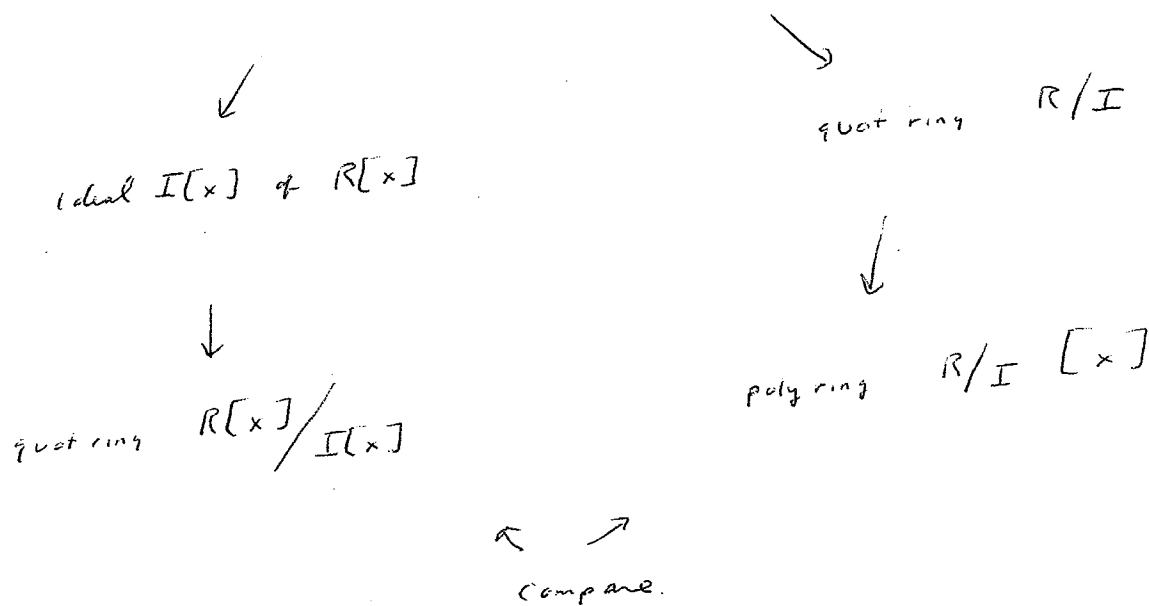
$$\sum_i a_i x^i \rightarrow \sum_i \varphi(a_i) x^i$$

• φ is surj iff $\hat{\varphi}$ is surj

• Let $I = \ker(\varphi)$. Then

$$\ker(\hat{\varphi}) = I[x]$$

Given ideal I of R



LEM 3 The following rings are isomorphic:

$$R[x]/I[x],$$

$$R/I[x]$$

pf Write $S = R/I$

Consider quotient map

$$\begin{array}{ccc}
 \psi & R & \longrightarrow S \\
 & r & \longrightarrow r+I
 \end{array}$$

ψ is surj with kernel I .

Now

$$\hat{\varphi} : R[x] \rightarrow S[x]$$

is surj with $\ker I[x]$.

Now $\hat{\varphi}$ induces ring iso

$$R[x]/I[x] \longrightarrow S[x]$$

$$f + I[x] \longrightarrow \hat{\varphi}(f)$$

Result follows.

□

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LEM 4 Given an ideal I of R
Consider ideal $I[x]$ of $R[x]$. TFAE

(i) I is prime

(ii) $I[x]$ is prime.

pt I prime

$\Leftrightarrow R/I$ is integral domain by LEM 1

$\Leftrightarrow R/I[x]$ is int. domain by LEM 3

$\Leftrightarrow R[x]/I[x]$ is int. domain.

$\Leftrightarrow I[x]$ is prime. \square

Given mutually commuting indeterminates

$$x_1, x_2, \dots, x_n$$

Define

$R[x_1, x_2, \dots, x_n]$ = ring of polynomials in x_1, x_2, \dots, x_n that have all coeff in R

View

$$R[x_1, x_2, \dots, x_n] = R[x_1, \dots, x_{n-1}][x_n]$$

A monomial in $R[x_1, \dots, x_n]$ has form

$$x_1^{e_1} x_2^{e_2} \dots x_n^{e_n} \quad e_i \geq 0 \quad (1 \leq i \leq n)$$

This monomial has

degree sequence (e_1, e_2, \dots, e_n)

total degree
(or just degree) $e_1 + e_2 + \dots + e_n$

Each $f \in R[x_1, \dots, x_n]$ is a finite sum

$$f = \sum_{i=1}^l a_i f_i \quad l \geq 0$$

where

$$a_i \neq 0 \in R \quad (1 \leq i \leq l),$$

f_1, f_2, \dots, f_l are mut dist monomials

Define

$$\deg(f) = \max \{ \deg(f_i) \mid 1 \leq i \leq l \}$$

Call f homogenous of degree k whenever

$$\deg(f_i) = k \quad 1 \leq i \leq l$$

• Each element of $R[x_1, \dots, x_n]$ is a sum of homogenous elements.

Given a field F

We have seen $F[x]$ is Euclidean domain.

We now consider this in more detail.

LEM 5 Given field F

Given $a, b \in F[x]$ with $b \neq 0$

Then \exists unique pair $q, r \in F[x]$ st

$$a = bq + r$$

and

$$r = 0 \quad \text{or} \quad \deg(r) < \deg(b)$$

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pf Show q, r exist.

Call the ordered pair a, b a counterexample whenever q, r do not exist.

Assume a, b is a counterexample.

obs $\deg(a) \geq \deg(b)$

else $*$, $**$ satisfied by

$$r = a; \quad q = 0$$

Among all counterexamples, WLOG assume

$$\deg(a) - \deg(b) \text{ is minimal.}$$

Write

$$a = a_0 + a_1x + \dots + \underset{\substack{\neq \\ 0}}{a_r}x^r$$

$$a_i \in F$$

$$b = b_0 + b_1x + \dots + \underset{\substack{\neq \\ 0}}{b_r}x^r$$

$$b_i \in F$$

$$0 \leq i \leq r$$

$b_r \neq 0$, so b^{-1} exists in F

define $A = a - b a_r^{-1} b_r x^{r-a} \in F[x]$

By constr: for A coef of x^r is 0

A does not involve $x^r, x^{r+1}, x^{r+2}, \dots$

so $A = 0$ or $\deg(A) < r = \deg(a)$

So A, b is not a counterexample.

So $\exists \varphi, r \in F[x]$ st

$$A = b\varphi + r$$

and

$$r = 0 \quad \text{or} \quad \deg(r) < \deg(b)$$

We have

$$A = b q + r$$

"

$$a = b a_1 b_2^{-1} x^{r-1} + r$$

$$a = b \left(\underbrace{q + a_1 b_2^{-1} x^{r-1}}_{\text{call this } q'} \right) + r$$

Now q, r satisfy $\star, \star\star$.

So a, b not a counterexample. cont.

We have shown q, r exist.

Next we show q, r is unique.

Suppose $\exists q', r' \in F[x]$ st

$$a = b q' + r'$$

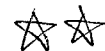
and

$$r' = 0 \quad \text{or} \quad \deg(r') < \deg(b)$$

show $q = q'$ and $r = r'$. Assume $q \neq q'$ else done.

Combine $\star, \star\star$ to get

$$b(q - q') = r' - r$$



So

$$\deg(b) + \deg(q - q') = \deg(r' - r)$$

we have

$$\deg(b) + \deg(q - q') = \deg(r' - r) < \deg(b)$$

by **, **

IV
deg(b)

cont.

So

$$q = q'$$

and

$$r = r'$$

□