Spring 2016

Math 542

Moden Algebra

MWF 8:50 AM

Ingraham 222

Lecture 1 Wed Jan 20

Our text is

Abstract Algebra 3d Ed by Dummit and Foote

In 541 we covered group theory and parts of ring theory. In 542 we cover more ring theory and module theory. (might be overlap at first)

We start with

Section 7.5: Rings of Fractions

Given a commutative ring R (perhaps without 1)

For $x \in R$

call x a <u>0-divisor</u> whenever

(i) $x \neq 0$,

(ii) $\exists \; 0 \neq y \in R$ s.t. $xy = 0$.

Given a nonempty subset $\mathcal{D} \subseteq R$ such that

- $0 \notin \mathcal{D}$
- $xy \in \mathcal{D} \quad \forall\, x, y \in \mathcal{D}$
- no element of $\mathcal{D}$ is a 0-divisor

We use $R, \mathcal{D}$ to construct a ring Q

Define

$F =$ set of ordered pairs

$(a, b) \qquad a \in R, \quad b \in \mathcal{D}$

Define a binary relation $\sim$ on F:

$(a, b) \sim (c, d)$ whenever $ad = bc$

LEM 1 $\sim$ is an equivalence relation

pf

$\sim$ is reflexive: $(a,b) \sim (a,b)$ ✓

$\sim$ is symmetric: $(a,b) \sim (c,d)$ iff $(c,d) \sim (a,b)$ ✓

$\sim$ is transitive:

$(a,b) \sim (c,d)$, $(c,d) \sim (e,f)$ implies $(a,b) \overset{?}{\sim} (e,f)$

$ad = bc$ $\quad$ $cf = de$ $\quad$ $af \overset{?}{=} be$

$adf = bcf = bde$

$d(af - be) = 0$

$d \in \mathcal{D}$ is not a 0-divisor

$af - be = 0$

$af = be$ ✓

$\square$

For $(a,b) \in F$ let

$\frac{a}{b}$ denote the equivalence class of $\sim$

that contains (a,b).

By definition,

$$\frac{a}{b} = \frac{c}{d} \quad \text{iff} \quad ad = bc$$

Define

$$\mathbb{Q} = \text{set of equiv classes of } \sim$$

LEM 2 Given $\frac{a}{b}, \frac{A}{B}, \frac{c}{d}, \frac{C}{D} \in \mathbb{Q}$ s.t.

$$\frac{a}{b} = \frac{A}{B} \qquad \frac{c}{d} = \frac{C}{D}$$

then

$$\frac{ad + bc}{bd} = \frac{AD + BC}{BD}$$

pf We have

$$aB = bA, \qquad cD = dC$$

Show

$$(ad+bc)BD \stackrel{?}{=} bd(AD+BC)$$

$$(ad+bc)BD = adBD + bcBD$$

$$bd(AD+BC) = bdAD + bdBC$$

$$adBD = aBdD = bAdD = bdAD$$

$$bcBD = bBcD = bBdC = bdBC$$

ok

$\square$

DEF 3 For $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$

define their sum to be

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

(this sum is well defined by LEM 2)

——— o ———

Note For $x, y \in \mathbb{Q}$

$$x + y = y + x$$

pf. By construction. $\square$

——— o ———

Observe

$$\frac{0}{b} = \frac{0}{B} \qquad \forall\ b, B \in \mathcal{D}$$

this common equivalence class will be denoted by $\mathbb{O}$.

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LEM 4 $\mathbb{Q}, +, \mathbb{0}$ is an abelian group

pf check axioms

<u>Assoc</u> $\forall x, y, z \in \mathbb{Q}$

$$(x+y) + z \overset{?}{=} x + (y+z)$$

Write

$$x = \frac{a}{b} \qquad y = \frac{c}{d} \qquad z = \frac{e}{f}$$

$(x+y)+z$ and $x+(y+z)$ both equal

$$\frac{adf + bcf + bde}{bdf}$$ ✓

<u>Ident</u> $\forall x \in \mathbb{Q},$

$$\mathbb{0} + x \overset{?}{=} x$$

write

$$\mathbb{0} = \frac{0}{b} \qquad x = \frac{c}{d}$$

$$\mathbb{0} + x = \frac{0 \cdot d + bc}{bd} = \frac{bc}{bd} = \frac{c}{d} = x$$

<u>Inv</u> Given $x \in \mathbb{Q}$, show $\exists\, y \in \mathbb{Q}$ s.t

$$x + y = \mathbb{0}$$

Write

$$x = \frac{a}{b}$$

$\mathbb{R}, +, 0$ is a group

$-a$ exists in $\mathbb{R}$

Define

$$y = \frac{-a}{b}$$

Obs

$$x + y = \frac{ab + b(-a)}{b^2} = \frac{0}{b^2} = \mathbb{0}$$

We have shown $\mathbb{Q}, +, \mathbb{0}$ is a group.

We saw earlier

$$x + y = y + x \quad \forall\, x, y \in \mathbb{Q}$$

So the group is abel. $\square$

LEM 5 Given $\frac{a}{b}, \frac{A}{B}, \frac{c}{d}, \frac{C}{D} \in \mathbb{Q}$ s.t.

$$\frac{a}{b} = \frac{A}{B}, \qquad \frac{c}{d} = \frac{C}{D}$$

then

$$\frac{ac}{bd} = \frac{AC}{BD}$$

pf We have

$$aB = bA, \qquad cD = dC$$

Show

$$acBD \stackrel{?}{=} bdAC$$

$$\begin{aligned} acBD &= aBcD \\ &= bAdC \\ &= bdAC \end{aligned}$$

✓

□

DEF 6 Given $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$,

define their product to be

$$\frac{a}{b} \frac{c}{d} = \frac{ac}{bd}$$

(the product is well defined by LEM 5)

— o —

LEM 7 For $x, y \in \mathbb{Q}$,

$$xy = yx$$

pf By construction. $\square$

— o —

Observe

$$\frac{b}{b} = \frac{B}{B} \qquad \forall\, b, B \in \mathcal{D}$$

This common equiv class will be denoted by $\mathbb{1}$

Note that $\mathbb{1} \neq \mathbb{O}$

LEM 8 The above data turns $\mathbb{Q}$ into a commutative ring with identity $\mathbb{1}$.

pf Check axioms

ASSOC $\forall\ x, y, z \in \mathbb{Q}$,

$$(xy)z \overset{?}{=} x(yz)$$

Write

$$x = \frac{a}{b} \qquad y = \frac{c}{d} \qquad z = \frac{e}{f}$$

$(xy)z$ and $x(yz)$ both equal

$$\frac{ace}{bdf}$$

DIST $\forall\ x, y, z \in \mathbb{Q}$

$$x(y+z) \overset{?}{=} xy + xz$$

Write

$$x = \frac{a}{b} \qquad y = \frac{c}{d} \qquad z = \frac{e}{f}$$

$$y + z = \frac{cf + de}{df}$$

$$x(y+z) = \frac{a(cf+de)}{bdf} = \frac{acf + ade}{bdf}$$

$$xy = \frac{ac}{bd}$$

$$xz = \frac{ae}{bf}$$

$$xy + xz = \frac{acbf + bdae}{b^2df}$$

$$= \frac{b(acf + ade)}{b(bdf)}$$

$$= \frac{acf + ade}{bdf}$$

Similarly

$$(x+y)z = xz + yz$$

IDENT $\forall x \in \mathbb{Q}$,

$$\mathbb{1}\, x \overset{?}{=} x$$

write $\mathbb{1} = \frac{b}{b}$ $x = \frac{c}{d}$

$$\mathbb{1}\, x = \frac{bc}{bd} = \frac{c}{d} = x$$

Similarly

$$x\, \mathbb{1} = x$$

We have turned $\mathbb{Q}$ into a ring with ident $\mathbb{1}$.

We saw earlier

$$xy = yx \quad \forall x, y \in \mathbb{Q}$$

So the ring is commutative. $\square$

We now consider how are R, Q related.

LEM 9 There exists an injective ring homomorphism

$$i : R \to Q$$

that sends

$$a \to \frac{ab}{b} \qquad \forall a \in R, \ \forall b \in \mathcal{D}$$

pf For $b \in \mathcal{D}$

$$i(0) = \frac{0b}{b} = \frac{0}{b} = 0$$

For $a, b \in R$,

$$i(a+b) \stackrel{?}{=} i(a) + i(b)$$

$$i(a) + i(b) = \frac{ac}{c} + \frac{bd}{d} \qquad c, d \in \mathcal{D}$$

$$= \frac{acd + bcd}{cd}$$

$$= \frac{(a+b)cd}{cd}$$

$$= i(a+b) \qquad \checkmark$$

For $a, b \in R$,

$$i(ab) \overset{?}{=} i(a)\, i(b)$$

$$i(a)\, i(b) = \frac{ac}{c} \quad \frac{bd}{d}$$

$$= \frac{abcd}{cd}$$

$$= i(ab) \qquad \checkmark$$

Show i is injective:

Given $a \in R$ such that $i(a) = \mathbb{O}$

Show $a = 0$

$$i(a) = \mathbb{O}$$

$$\frac{ab}{b} \qquad \frac{0}{d} \qquad b, d \in \mathcal{D}$$

$$abd = b0 = 0$$

$bd \in \mathcal{D}$ is not a 0-divisor

So $a = 0$

$\square$