

Combinatorics 475

Lecture 9

LEC9 1

We continue to discuss the inversion sequence of a permutation.

LEM Given a perm of $\{1, 2, \dots, n\}$

with inversion sequence $(b_1, b_2, \dots, b_n)$

then

$$0 \leq b_1 \leq n-1$$

$$0 \leq b_2 \leq n-2$$

$$0 \leq b_3 \leq n-3$$

$$\vdots$$

$$0 \leq b_{n-1} \leq 1$$

$$0 = b_n$$

pf For $1 \leq i \leq n$ show

$$0 \leq b_i \leq n-i.$$

call the perm $a_1 a_2 \dots a_n$

$b_i = \#$ elements among $a_1, a_2, \dots, a_n$ that are larger than i
and to left of i

$\leq \#$ elements among $a_1, a_2, \dots, a_n$ that are larger than i

$$\left[\text{choices are } \underbrace{i+1, i+2, \dots, n-1, n}_{n-i \text{ poss}} \right]$$

$$\leq n-i$$

□

We can recover a perm of $\{1, 2, \dots, n\}$ from its inversion sequence as follows.

ex $n=6$

i	1	2	3	4	5	6
b_i	4	0	2	1	1	0

Find orig perm $a_1 a_2 a_3 a_4 a_5 a_6$

Method 1

6
6 5
6 4 5
6 4 3 5
2 6 4 3 5
2 6 4 3 1 5

Method 2

—	—	—	—	<u>1</u>	—
<u>2</u>	—	—	—	<u>1</u>	—
<u>2</u>	—	—	<u>3</u>	<u>1</u>	—
<u>2</u>	—	<u>4</u>	<u>3</u>	<u>1</u>	—
<u>2</u>	—	<u>4</u>	<u>3</u>	<u>1</u>	<u>5</u>
<u>2</u>	<u>6</u>	<u>4</u>	<u>3</u>	<u>1</u>	<u>5</u>

In summary we have

Thm: Given an integer $n \geq 1$

the function which sends a permutation of $\{1, 2, \dots, n\}$ to its inversion sequence is a bijection from the set of all perms of $\{1, 2, \dots, n\}$ to the set

$$\{0, 1, \dots, n-1\} \times \{0, 1, \dots, n-2\} \times \dots \times \{0, 1, 2\} \times \{0, 1\} \times \{0\}. \quad (*)$$

pf Cardinality of set $(*)$ is $n!$

$$= \# \text{ perms of } \{1, 2, \dots, n\}$$

So Suffices to show function is 1-1.

function is 1-1 since each perm of $\{1, 2, \dots, n\}$ is determined by its inversion sequence. $\square$

Given a perm $a_1 a_2 \dots a_n$ of $\{1, 2, \dots, n\}$
 with inversion sequence $(b_1, b_2, \dots, b_n)$

Recall

$$b_1 + b_2 + \dots + b_n = \text{total \# of inversions}$$

"inversion number"

"length"

the inversion number
 is equal to the minimum number of transpositions required
 to bring $a_1 a_2 \dots a_n$ to $123 \dots n$.

Reason: let $l =$ inversion number of $a_1 a_2 \dots a_n$

At most l transpositions required:

use b_1 transpositions to move "1" to position 1
 $1 \times \dots \times$

use b_2 "2" to position 2
 $12 \times \dots \times$

use b_n "n" to position n
 $12 \dots n$

At least l transpositions required: each transposition
 changes the inversion number by 1 (± 1)

□

ex $n=6$

perm 624135

inversion require

i	1	2	3	4	5	6
b_i	3	1	2	1	1	0

$$\text{Inversion number} = 3 + 1 + 2 + 1 + 1 + 0$$

$$= 8$$

Bring 624135 to 123456 using 8 transpositions:

perm	to do	# transpositions used
624135	move "1"	
X 621435		
X 612435		3
X 162435		
X 126435	move "2"	1
X 126345	move "3"	2
X 123645	move "4"	1
X 123465	move "5"	1
X 123456	move "6"	0

4.3 Generating combinations

Problem. Given a set S with $|S|=n$

List all the subsets of S

Ex $n=3$ $S = \{1, 2, 3\}$

One List is

$$\emptyset \quad \{1\} \quad \{2\} \quad \{3\} \quad \{12\} \quad \{13\} \quad \{23\} \quad \{123\}$$

For large n gets complicated

We now present a Listing method with nice properties

From now on take

$$S = \{x_{n-1}, x_{n-2}, \dots, x_2, x_1, x_0\}$$

Step 1

We identify each subset Ω of S with a sequence

$a_{n-1} a_{n-2} \dots a_1 a_0$ of zeros and ones:

For $0 \leq i \leq n-1$

$$a_i = \begin{cases} 1 & \text{if } x_i \in \Omega \\ 0 & \text{if } x_i \notin \Omega \end{cases}$$

ex $n=6$

$$S = \{x_5, x_4, x_3, x_2, x_1, x_0\}$$

Take

$$\Omega = \{x_4, x_2, x_1\}$$

x_5	x_4	x_3	x_2	x_1	x_0
0	1	0	1	1	0

Step 2

View each sequence of zeros and ones
as the "base 2" representation of a
non-negative integer

ex the sequence 010110 corresponds to the integer

$$\begin{aligned}
 & 0 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 \\
 & = \\
 & \quad 16 + 4 + 2 \\
 & = 22
 \end{aligned}$$

ex Find the "base 2" representation of 57

Sol

Recall

i	5	4	3	2	1	0
2^i	32	16	8	4	2	1

take out 32

$$\begin{array}{r} 57 \\ - 32 \\ \hline 25 \end{array}$$

take out 16

$$\begin{array}{r} 25 \\ - 16 \\ \hline 9 \end{array}$$

take out 8

$$\begin{array}{r} 9 \\ - 8 \\ \hline 1 \end{array}$$

So $57 =$

answer =

term	2^5	2^4	2^3	2^2	2^1	2^0
coef	1	1	1	0	0	1

111001

Step 3

the list

$$0, 1, 2, 3, \dots, 2^n - 1$$

of integers, each expressed in base 2,

effectively gives

all the subsets

of an n -element set

"squashed order"

ex $n=3$

m	m in base 2	corresp subset of $\{x_2, x_1, x_0\}$
0	000	$\emptyset$
1	001	$\{x_0\}$
2	010	$\{x_1\}$
3	011	$\{x_1, x_0\}$
4	100	$\{x_2\}$
5	101	$\{x_2, x_0\}$
6	110	$\{x_2, x_1\}$
7	111	$\{x_2, x_1, x_0\}$

← squashed order

ex For $n=7$

Consider subset

$$\{x_5, x_4, x_2, x_1, x_0\}$$

f

$$S = \{x_6, x_5, x_4, x_3, x_2, x_1, x_0\}$$

Find the next subset in the squashed order.

Sol. Convert to base 2:

x_6	x_5	x_4	x_3	x_2	x_1	x_0
0	1	1	0	1	1	1

add 1:

0	1	1	0	1	1	1
						1
+						
0	1	1	1	0	0	0

Convert back:

$$\{x_5, x_4, x_3\}$$

□

