Math 846

Lecture 9

Earlier we gave four examples of DRGs.

We now give another type of example.

For the moment let $\Gamma = (X, \mathcal{R})$ be any graph.

An <u>automorphism</u> of Γ is a bijection $\sigma : X \to X$ that leaves $\mathcal{R}$ invariant. So $\forall x, y \in X$,

x, y adjacent iff $\sigma(x), \sigma(y)$ adjacent

The set $\mathrm{Aut}(\Gamma)$ of automorphisms of Γ forms a group under composition.

Ex 5 For the complete graph $\Gamma = K_n$
$\mathrm{Aut}(\Gamma)$ is the symmetric group S_n

<u>Note</u> For a graph $\Gamma = (X, \mathcal{R})$ and $\sigma \in \mathrm{Aut}(\Gamma)$ for notational convenience we often view σ as a permutation matrix in $\mathrm{Mat}_X(\mathbb{F})$. From this viewpoint

$$\mathrm{Aut}(\Gamma) = \left\{ P \in \mathrm{Mat}_X(\mathbb{F}) \;\middle|\; \begin{array}{l} P \text{ is a permutation matrix and} \\ PA = AP \end{array} \right\}$$

Given a graph $\Gamma = (X, \mathcal{R})$

Any subgroup $G \subseteq \mathrm{Aut}(\Gamma)$ acts on the set X.

type of action	meaning
transitive	$\forall x, y \in X \quad \exists\, \sigma \in G$ st $\sigma(x) = y$
regular	$\forall x, y \in X \quad \exists$ unique $\sigma \in G$ st $\sigma(x) = y$
generously transitive	$\forall x, y \in X \quad \exists\, \sigma \in G$ st $\sigma(x) = y$ and $\sigma(y) = x$
distance-transitive	$\forall\, x, y, x', y' \in X$ st $\partial(x,y) = \partial(x', y')$, $\exists\, \sigma \in G$ st $\sigma(x) = x'$ and $\sigma(y) = y'$

LEM 6 For a connected graph $\Gamma = (X, \mathcal{R})$, assume $\mathrm{Aut}(\Gamma)$ is distance-transitive on X. Then Γ is distance-regular.

pf ex. $\square$

<u>Note</u> All the graphs in Ex 3 are distance-trans.

Next goal: show any DRG with diameter D has exactly $D+1$ dist eigenvalues.

LEM 7 For a DRG $\Gamma = (X, R)$ with diameter D

$$A A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \qquad (0 \le i \le D)$$

where $b_{-1} = 0$, $c_{D+1} = 0$ and we recall $A_{-1} = 0$, $A_{D+1} = 0$

pf For $x, y \in X$

$$(A A_i)_{xy} = \sum_{z \in X} A_{xz} (A_i)_{zy}$$

$$= \sum_{z \in \Gamma(x) \cap \Gamma_i(y)} 1$$

$$= | \Gamma(x) \cap \Gamma_i(y) |$$

$$= \begin{cases} b_{i-1} & \text{if } \partial(x,y) = i-1 \\ a_i & \text{if } \partial(x,y) = i \\ c_{i+1} & \text{if } \partial(x,y) = i+1 \end{cases}$$

= (x,y)-entry of

$$b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}$$

□

LEM 8 With above notation,

(i) $\{A_i\}_{i=0}^{D}$ form a basis for the adjacency algebra M

(ii) the $\mathbb{F}$-vector space M has dimension $D+1$

(iii) Γ has exactly $D+1$ distinct eigenvalues

pf (i) Recall $\{A_i\}_{i=0}^{D}$ are linearly indep. Define

$M' = \mathrm{Span}\{A_i\}_{i=0}^{D}$. We have

$AM' \subseteq M'$ by LEM 7 and $I = A_0 \in M'$

so $M \subseteq M'$

Also $\dim M' = D+1$ and $\dim M \geq D+1$ so $M = M'$

(ii), (iii) By (i)

EX Given a DRG $\Gamma = (X, \mathcal{R})$ and a subgroup $G \subseteq \mathrm{Aut}(\Gamma)$ that acts distance-transitively on X, the adjacency algebra

$$M = \{ B \in \mathrm{Mat}_X(\mathbb{F}) \mid B\rho = \rho B \quad \forall \rho \in G \}.$$

LEM 9 Given a DRG $\Gamma = (X, \mathcal{R})$ with diameter D.

then for $0 \le j \le D$,

(i) $E_j V$ is a module for $\mathrm{Aut}(\Gamma)$

(ii) Suppose $G \subseteq \mathrm{Aut}(\Gamma)$ is distance-trans on X.
then the G-module $E_j V$ is irreducible.

pf (i) For $P \in \mathrm{Aut}(\Gamma)$,

$$PB = BP \quad \forall B \in M$$

So $$P E_j = E_j P$$

So $$P E_j V = E_j PV = E_j V$$

(ii) Given $0 \ne W \subseteq E_j V$ st $GW \subseteq W$.

show $W = E_j V$

Pick $0 \ne w \in W$

Write $$w = \sum_{x \in X} w_x \hat{x} \qquad w_x \in \mathbb{F}$$

$\exists\, x \in X \quad \text{st} \quad w_x \neq 0$

Consider subgroup

$$\text{Stab}(x) \leq G$$

$\text{Stab}(x)$ is transitive on $\Gamma_i(x)$ for $0 \leq i \leq D$

Define

$$v = \frac{1}{|\text{Stab}(x)|} \sum_{\rho \in \text{Stab}(x)} \rho w$$

$$\in W$$

Write $\displaystyle v = \sum_{y \in X} v_y \hat{y}$ $\qquad v_y \in \mathbb{F}$

$$v_x = w_x \neq 0$$

so $v \neq 0$

We have

$$v = \sum_{y \in X} v_y \hat{y}$$

$$= \sum_{i=0}^{D} \sum_{y \in \Gamma_i(x)} v_y \hat{y}$$

depends only on i
call it α_i

$$= \sum_{i=0}^{D} \alpha_i \sum_{y \in \Gamma_i(x)} \hat{y}$$

$$= \underbrace{\sum_{i=0}^{D} \alpha_i A_i}_{B} \hat{x}$$

$B \in M$, write $B = \sum_{l=0}^{D} \gamma_l E_l \qquad \gamma_l \in \mathbb{F}$

$v \in W \subseteq E_j V$ so $v = E_j v$

Now

$$v = E_j v = E_j B \hat{x} = \gamma_j E_j \hat{x}$$

$\gamma_j \neq 0$ since $v \neq 0$ so

$$E_j \hat{x} \in W$$

W is G-invariant and G is trans on X so

$$E_j \hat{y} \in W \qquad \forall y \in X$$

Obs

$$E_j V = \text{Span}(E_j \hat{y} \mid y \in X) \subseteq W \subseteq E_j V$$

So $\qquad W = E_j V$

$\square$

Next goal: how to compute the spectrum of a DRG from its intersection numbers.

<u>Notation</u> let λ = indeterminate

$\mathbb{F}[\lambda]$ = $\mathbb{F}$-algebra of all polynomials in λ that have coefficients in $\mathbb{F}$

DEF 10 Given a DRG $\Gamma = (X, R)$ of diameter D

Define polynomials $\{v_i\}_{i=0}^{D+1}$ in $\mathbb{F}[\lambda]$ by

$$v_0 = 1, \qquad v_1 = \lambda$$

$$\lambda v_i = c_{i+1} v_{i+1} + a_i v_i + b_{i-1} v_{i-1} \qquad 1 \le i \le D$$

where $c_{D+1} = 1$

LEM 11 With above notation,

(i) $\deg v_i = i$ $(0 \le i \le D+1)$

(ii) the coef of λ^i in v_i is $(c_1 c_2 \cdots c_i)^{-1}$ $(0 \le i \le D+1)$

(iii) $v_i(A) = A_i$ $(0 \le i \le D)$

(iv) $v_{D+1}(A) = 0$

(v) the dist eigenvalues of Γ are precisely the zeros of v_{D+1}.

pf

(i), (ii) By Def 10

(iii), (iv) Compare LEM 7, Def 10

(v) let $m \in \mathbb{F}[\lambda]$ denote the minimal polynomial of A.

ie monic poly with least degree such that $m(A) = 0$

Since A is diagonalizable, the distinct eigenvalues of A are precisely the roots of m. Each of m, v_{D+1} has degree $D+1$ and $v_{D+1}(A) = 0$ so v_{D+1} is a scalar mult of m. Result follows.

$\square$