

Combinatorics 475

Lecture 8

Ch 4 Generating Permutations and Combinations

4.1 Generating permutations

Problem: List all the permutations of $\{1, 2, \dots, n\}$

ex $n=3$

123
132
213
231
312
321

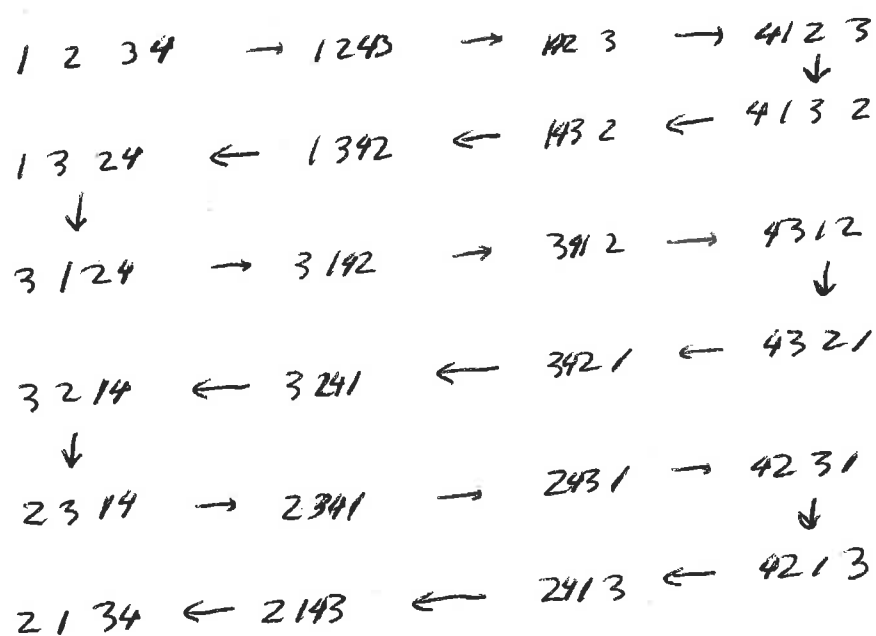
For large n , hard to keep track!

We now present a listing method with nice properties

$n=2$ 12
 21

$n=3$ In the $n=2$ list, insert 3 in all possible ways
as follows:

123
132
312
321
231
213

$n=4$  $n \geq 5$ similar

For $n \geq 2$ the list of permutations of $\{1, 2, \dots, n\}$ is obtained from the list of permutations of $\{1, 2, \dots, n-1\}$ by inserting " n " in all possible ways, as shown above.

Nice properties

Given a permutation $a_1 a_2 \dots a_n$ of $\{1, 2, \dots, n\}$

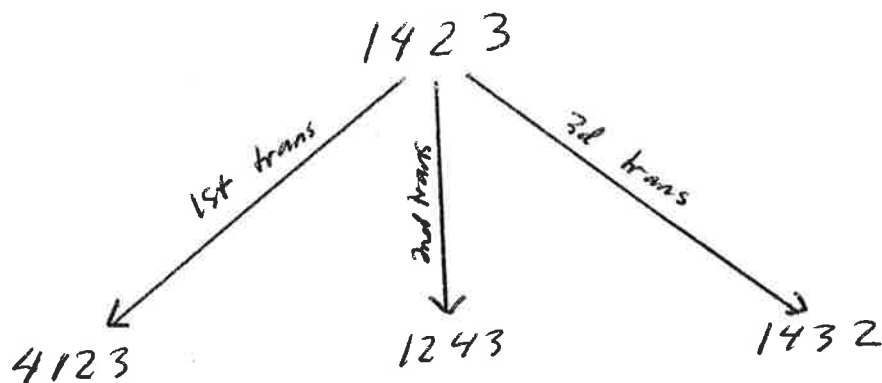
A transposition of $a_1 a_2 \dots a_n$ switches adjacent terms

$$\begin{array}{ccccccc} a_1 & a_2 & \dots & a_i & a_{i+1} & \dots & a_n \\ \downarrow & \downarrow & \dots & \downarrow & \downarrow & \dots & \downarrow \\ a_1 & a_2 & \dots & a_{i+1} & a_i & \dots & a_n \end{array}$$

"
ith transposition
"

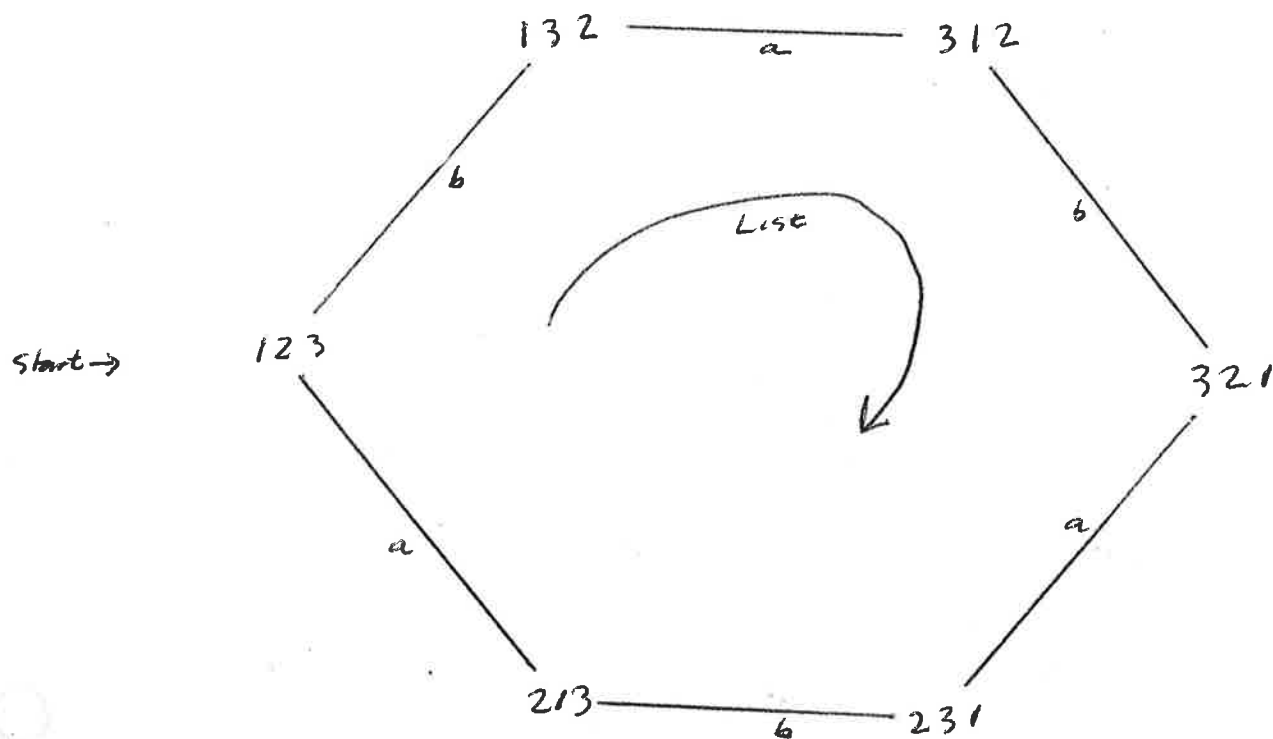
$a_1 a_2 \dots a_n$ has $n-1$ transpositions

ex $n=4$

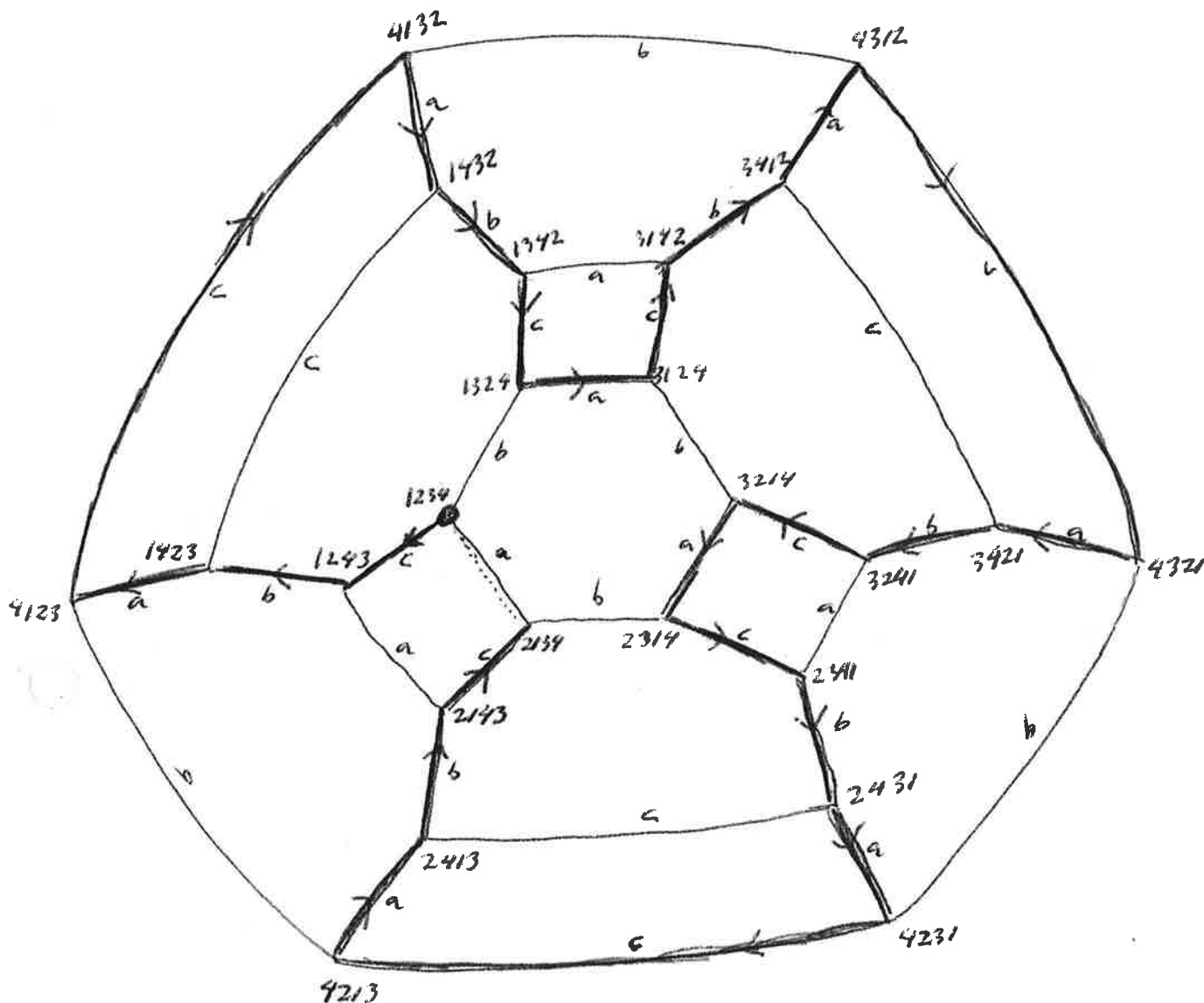


* In our List of perms of $\{1, 2, \dots, n\}$
 each perm in List is a transposition of
 the preceding perm. Also, the first
 perm is a transposition of Last perm.

ex $n=3$



$a = 1st \text{ transposition}$
 $b = 2nd \text{ transposition}$



Listing order
follows path

$a = \text{Kt transposition}$
 $b = \text{2nd } \dots$
 $c = \text{3d } \dots$

View as polytope in
3 dimensions

4.2 Inversions of Permutations

For an integer $n \geq 1$

there are $n!$ perms of $\{1, 2, \dots, n\}$

Consider Cartesian product

$$\{0, 1, \dots, n-1\} \times \{0, 1, \dots, n-2\} \times \dots \times \{0, 1, 2\} \times \{0, 1\} \times \{0\}$$

(*)

The set (*) has $n!$ elements.

Next goal: display a bijection between (*) and

the set of all perms of $\{1, 2, \dots, n\}$

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Given a perm $a_1, a_2, \dots, a_n$ of $\{1, 2, \dots, n\}$

an inversion of this perm is an ordered pair (a_k, a_l)

such that

$$k < l \quad \text{and} \quad a_k > a_l$$

" a_k and a_l are out of order "

ex $n=4$

perm: 4132

inversions: (4,1) (4,3) (4,2) (3,2)

Given a permutation $a_1 a_2 \dots a_n$ of $\{1, 2, \dots, n\}$ For $1 \leq i \leq n$ let
$$b_i = \# \text{ of inversions that have } i \text{ as 2nd coord}$$

$$= \# \text{ of elements among } a_1, a_2, \dots, a_n \text{ that are larger than } i \text{ and appear to left of } i$$
The sequence $(b_1, b_2, \dots, b_n)$ is called the inversion sequencefor $a_1 a_2 \dots a_n$.Note $b_1 + b_2 + \dots + b_n = \text{total number of inversions for } a_1 a_2 \dots a_n$ ex $n=4$ perm 4132

i	1	2	3	4
b_i	1	2	1	0

$$1 + 2 + 1 + 0 = 4 = \# \text{ inversions for } 4132$$

