Math 846

Lecture 8

LEM 31 Assume Γ is connected and regular with valency k. Then Γ has maximal eigenvalue k. The corresponding eigenspace has basis

$$\mathbb{1} = \sum_{x \in X} \hat{x}$$

pf We have

$$k = k_{min} \le \theta_0 \le k_{max} = k$$

So $\theta_0 = k$.

The k-eigenspace has dimension 1 since Γ is connected.

One checks $A \mathbb{1} = k \mathbb{1}$.

The result follows. $\square$

Assume our graph $\Gamma = (X, R)$ is regular with valency k.

Assume $k \geq 1$ so Γ has an edge.

Consider 2nd largest eigenvalue θ_1 of Γ.

LEM 32 Assume Γ is regular with valency $k \geq 1$.

Then

(i) $\theta_1 \geq -1$

(ii) $\theta_1 = -1$ iff Γ is a disjoint union of complete graphs.

pf By Cor 2 the spectrum of Γ satisfies

$$\sum_{i=0}^{r} m_i = |X|$$

$$\sum_{i=0}^{r} m_i \theta_i = 0$$

$$\sum_{i=0}^{r} m_i \theta_i^2 = |X| k$$

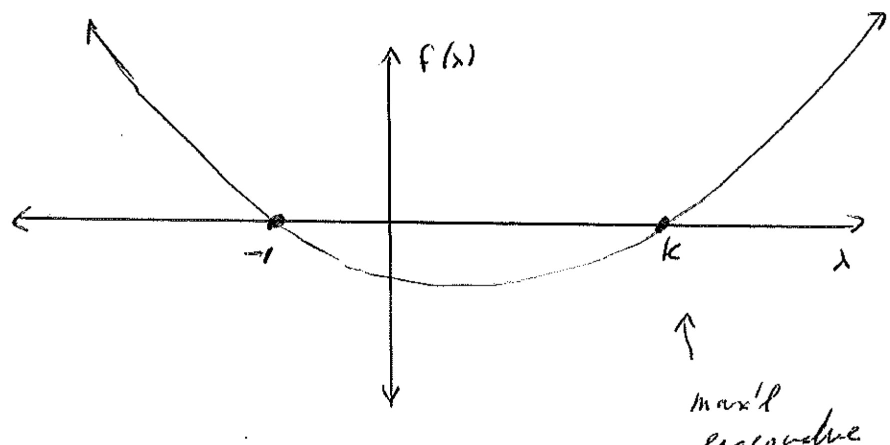
Define a polynomial in a variable λ:

$$f(\lambda) = (\lambda+1)(\lambda-k)$$

Obs

$$\sum_{i=0}^{r} m_i f(\theta_i) = |X|k - |X|k$$

$$= 0 \qquad \divideontimes$$



Suppose $\theta_1 < -1$. Then

$$f(\theta_i) > 0 \qquad (1 \le i \le r)$$

This contradicts $\divideontimes$

Suppose $\theta_1 = -1$ Then

$$f_i(\theta_i) > 0 \qquad (2 \le i \le r)$$

Contradiction unless $r = 1$.

So Γ has only eigenvalues k, -1

Γ is disjoint union of complete graphs.

$\square$

Exercise II

Find all the connected graphs Γ that have maximal eigenvalue < 2.

Chapter 2. Distance-regular graphs.

Lec 5

We now consider a class of graphs whose spectrum is well behaved. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Let $\Gamma = (X, R)$ denote a connected graph with diameter D. For $0 \le i \le D$ define $A_i \in \mathrm{Mat}_X(\mathbb{F})$ with (x,y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \ne i \end{cases} \qquad x, y \in X$$

Call A_i the <u>ith distance matrix</u> for Γ

For notational convenience put $A_{-1} = 0$, $A_{D+1} = 0$

Observations

- $\{A_i\}_{i=0}^{D}$ are linearly independent
- For $x \in X$ and $0 \le i \le D$,

$$A_i \hat{x} = \sum_{y \in \Gamma_i(x)} \hat{y}$$

- $A_0 = I$

- $A_1 = A$ (adjacency matrix)

- $\sum_{i=0}^{D} A_i = J$ (the all 1's matrix)

- $\overline{A_i} = A_i \qquad 0 \le i \le D$

- $A_i^t = A_i \qquad 0 \le i \le D$

DEF 1 The graph Γ is called <u>distance-regular</u> (or DRG) whenever for $0 \le i \le D$ and $x, y \in X$ at $\partial(x,y) = i$, the scalars

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|, \qquad a_i = |\Gamma_i(x) \cap \Gamma(y)|,$$

$$b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|$$

are independent of x, y and depend only on i.

We call c_i, a_i, b_i the <u>intersection numbers</u> of Γ.

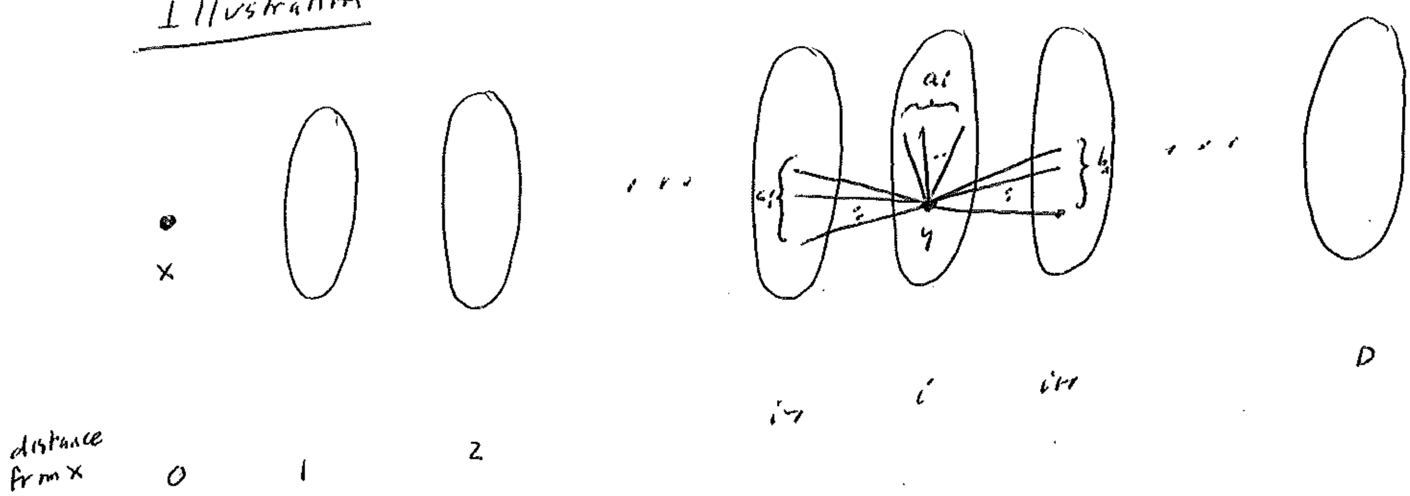
Observe

$$c_0 = 0, \qquad a_0 = 0, \qquad c_1 = 1, \qquad b_D = 0$$

$$c_i > 0 \qquad 1 \le i \le D$$

$$b_i > 0 \qquad 0 \le i \le D-1$$

Illustration — Fix $x, y \in X$ at $\partial(x,y) = i$



Ex The hypercube $H(D,2)$ is distance-regular with intersection numbers

$$c_i = i, \qquad a_i = 0, \qquad b_i = D - i \qquad (0 \le i \le D)$$

Lem 2 Assume Γ is Distance-Regular. Then

(i) Γ is regular with valency $k = b_0$

(ii) $k = c_i + a_i + b_i$ $(0 \le i \le D)$

(iii) $c_1 \le c_2 \le \cdots \le c_D$

(iv) $k > b_1 \ge b_2 \ge \cdots \ge b_{D-1}$

(v) $c_i \le b_j$ if $i + j \le D$ $(0 \le i, j \le D)$

pf (i) Set $i = 0$ in def of b_i

(ii) See above illustration.

(iii) – (v) Routine $\square$

There are many known ∞ families of DRGs. We mention a few families

Ex 3 For $D \in \mathbb{N}$ the following graphs are distance-regular with diameter D

(i) Hamming graph $H(D, N)$ $N \ge 2$

Fix a set S with $|S| = N$

X = set of all D-tuples of elements from S

vertices $x, y \in X$ are adjacent whenever x, y differ in exactly one coordinate

Here

$$c_i = i \qquad b_i = (D-i)(N-1) \qquad 0 \le i \le D$$

(ii) The Johnson graph $J(D,N)$ $\quad (N \geq 2D)$

Fix a set S with $|S| = N$

X = set of all D-element subsets of S

Vertices $x, y \in X$ are adjacent whenever

$$|x \cap y| = D-1$$

Here

$$c_i = i^2, \qquad b_i = (D-i)(N-D-i) \qquad 0 \le i \le D$$

(iii) The q-Johnson graph $J_q(D,N)$ $\quad (N \geq 2D)$

Also called Grassmann graph

Fix a finite field $\mathbb{F}_q$

Let W denote an N-dimensional vector space over $\mathbb{F}_q$

X = set of all D-dimensional subspaces of W

Vertices $x, y \in X$ are adjacent whenever

$$\dim(x \cap y) = D-1$$

Here

$$c_i = \left(\frac{q^i - 1}{q - 1} \right)^2 \qquad 0 \le i \le D$$

$$b_i = \frac{q^{2i+1} (q^{D-i} - 1)(q^{N-D-i} - 1)}{(q-1)^2} \qquad 0 \le i \le D$$

(iv) Bilinear forms graph $H_q(D, N)$ $(N \geq D)$

Fix a finite field $\mathbb{F}_q$

X = set of all $D \times N$ matrices with entries in $\mathbb{F}_q$.

Vertices $x, y \in X$ are adjacent whenever

$$\operatorname{rank}(x - y) = 1$$

Here

$$c_i = q^{i-1} \frac{q^i - 1}{q - 1} \qquad 0 \le i \le D$$

$$b_i = q^{2i} \frac{(q^{D-i} - 1)(q^{N-i} - 1)}{q - 1} \qquad 0 \le i \le D$$

$\square$

(v) The Odd graph O_{D+1}

X = set of all D-subsets of an $(2D+1)$-set.

Vertices $x, y \in X$ are adjacent whenever

$x \cap y = \phi$.

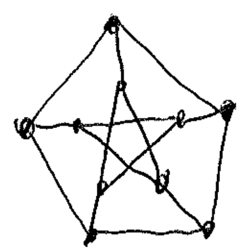
Here

$$c_i = \left\lfloor \frac{i+1}{2} \right\rfloor \qquad (0 \le i \le D)$$

$$b_i = D+1-c_i \qquad (0 \le i \le D-1)$$

Note that $a_i = 0$ for $0 \le i \le D-1$ and $a_D \ne 0$ "almost bipartite"

Note that O_3 is the Peterson graph



O_4 has intersection numbers

c_1	c_2	c_3	k	b_1	b_2
1	1	2	4	3	3

For a DRG $\Gamma = (X, R)$ with diameter D

Fix $x \in X$ and define

$$k_i = |\Gamma_i(x)| \qquad 0 \le i \le D$$

Obs $k_0 = 1$ and $k_1 = k$ = valency of Γ

LEM 4 With above notation,

(i) $k_i c_i = k_{i-1} b_{i-1} \qquad 1 \le i \le D$

(ii) $k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad 0 \le i \le D$

(iii) k_i is independent of x

pf (i) Each of $k_i c_i$, $k_{i-1} b_{i-1}$ is the number of edges between $\Gamma_{i-1}(x)$, $\Gamma_i(x)$

(ii) By (i)

(iii) By (ii) $\square$

We call k_i the <u>ith valency</u> of Γ