

Combinatorics 475

Lecture 7

3.3 Ramsey theory

We will be discussing the complete graph K_n

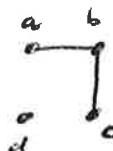
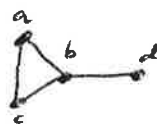
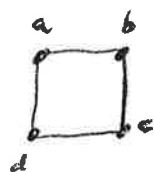
A graph is a pair (X, E) where

- (i) X is a finite set (the vertices)
- (ii) E is a set of 2-element subsets of X (the edges)

Vertices $x, y \in X$ are adjacent whenever $x \neq y$ and $xy \in E$.

When we draw a graph, adjacent vertices are connected by an arc.

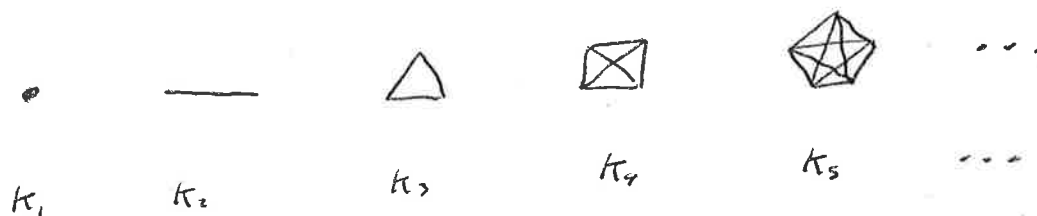
Ex Some graphs with vertex set $X = \{a, b, c, d\}$:



$$E = \{ab, bc, cd, da\}$$

Graph on right is complete Complete means any pair of distinct vertices are adjacent

Let K_n denote the complete graph on n vertices.



K_n has $\binom{n}{2}$ edges

Def. Given a graph (X, E)

By a subgraph of (X, E) we mean a graph (X', E') where

$$X' \subseteq X$$

$$E' = \{xy \mid x, y \in X', xy \in E\}$$

LEM For the complete graph K_6

Suppose we color each edge red or blue.

Then either:

(i) There exists a K_3 subgraph with all edges red
"a red K_3 "

or

(ii) There exists a K_3 subgraph with all edges blue
"a blue K_3 "

pf We assume (i) fails and show (ii)

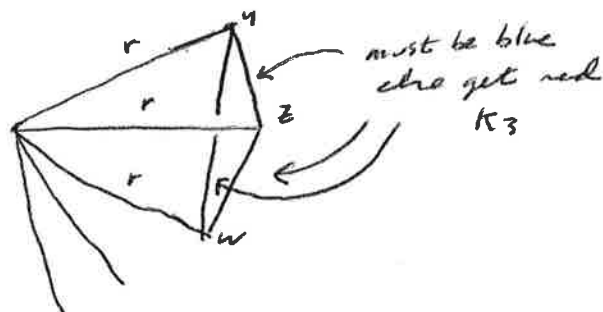
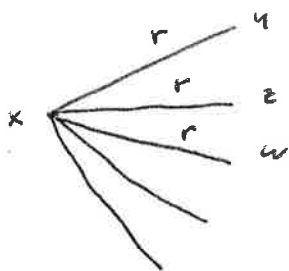
Pick a vertex x of K_6

5 edges contain x

By Pigeonhole principle either

• at least 3 of these edges are red
or
• at least 3 of these edges are blue

wlog at least 3 are red



Now the subgraph with vertices y, z, w is a blue K_3

□

Def Given an integer $t \geq 2$

$$\text{For } 2 \leq m, n \leq t$$

$$K_t \rightarrow K_m, K_n$$

means:

No matter how we color the edges of K_t red or blue,
there exists a red K_m subgraph or a blue K_n subgraph

Ref to above Def. obs

$$K_t \rightarrow K_m, K_n$$

implies

$$K_s \rightarrow K_m, K_n$$

for all integers $s \geq t$

Problem Given integers $m, n \geq 2$

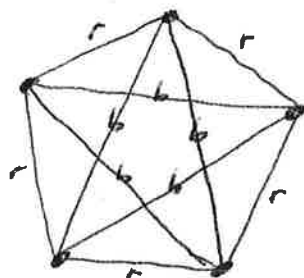
- Does $K_t \rightarrow K_m, K_n$ for sufficiently large t ?
- If so, find the minimal t such that $K_t \rightarrow K_m, K_n$.

This minimal t is called the Ramsey number $r(m, n)$

$$\text{obs } r(m, n) = r(n, m)$$

LEM $r(3,3) = 6$ pf We showed $K_6 \rightarrow K_3, K_3$ Also $K_5 \not\rightarrow K_3, K_3$

Reason:



□

LEM $r(n,2) = n$ $n = 2, 3, \dots$

pf Routine

□

thm Given integers $m, n \geq 2$. Then

$$K_t \rightarrow K_m, K_n$$

" $r(m, n)$ exists"

for suf large t .

pf

By induction on

$m+n$

We may assume $m \geq 3, n \geq 3$; else we are done by prev lemma.

By induction

$r(m, n-1)$ exists

and

$r(m-1, n)$ exists.

define

$$t = r(m, n-1) + r(m-1, n)$$

show

$$K_t \rightarrow K_m, K_n$$

Color each edge of K_t red or blue

Pick vertex x of K_t

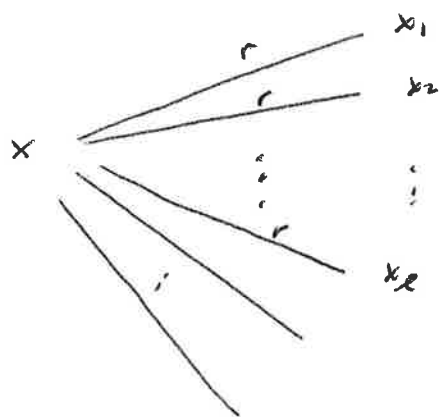
$t-1$ edges of K_t contain x

By PP either

(i) at least $r(m-1, n)$ of these edges are red

or
(ii) at least $r(m, n-1)$ of these edges are blue

Assume Case (i): the other case similar.



$$l = r(m-1, n)$$

By def

$$K_l \rightarrow K_m, K_n$$

Consider K_l subgraph with vertices $x_1, x_2, \dots, x_l$.

Either:

- it has a red K_m subgraph
 [which together with x forms a red K_m subgraph of K_t]

or

- it has a blue K_n subgraph
 [which is a blue K_n subgraph of K_t]

Therefore K_t has a red K_m subgraph or a blue K_n subgraph.

So

$$K_t \rightarrow K_m, K_n$$

□

the above thm shows that

$r(m, n)$ exists for all integers $m, n \geq 2$.

thm For integers $m, n \geq 2$

$$r(m, n) \leq \binom{m+n-2}{m-1}$$

pf By induction on $m+n$

Case $m=2$

$$r(2, n) = n = \binom{n}{1} \quad \checkmark$$

Case $n=2$

$$r(m, 2) = m = \binom{m}{m-1} \quad \checkmark$$

Case $m \geq 3, n \geq 3$

define

$$t = r(m, n-1) + r(m-1, n)$$

By the proof of the prev thm

$$K_t \rightarrow K_m, K_n$$

So

$$r(m, n) \leq t$$

Now

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$$r(m, n) \leq t$$

$$= r(m, n-1) + r(m-1, n)$$

by ind

$$\leq \binom{m+n-2}{m-1} + \binom{m+n-2}{m-1}$$

$$= \binom{m+n-3}{m-1} + \binom{m+n-3}{m-2}$$

$$= \binom{m+n-2}{m-1}$$

□

