Math 846
Lecture 7

A vector $v \in \mathbb{R}^X$ is Frobenius whenever the entries of v have the same sign

(all > 0 or all < 0).

Thm 27 (Perron - Frobenius) Given $B \in \mathrm{Mat}_X(\mathbb{R})$

Assume

- B is symmetric
- B is irreducible
- $B_{xy} \geq 0 \quad \forall x, y \in X$.

Let θ_0 = max'l eigenvalue of B.

Then

(i) every θ_0-eigenvector for B is Frobenius

(ii) the θ_0-eigenspace for B has dimension 1

(iii) No other eigenspace of B contains a Frobenius vector

(iv) $\theta_0 \geq 0$

(v) every eigenvalue of B is at least $-\theta_0$

(vi) B has eigenvalue $-\theta_0$ iff B is bipartite.

pf (i) Obs

$$\theta_0 I - B \quad \text{is pos semi def}$$

View $\theta_0 I - B$ as inner product matrix for some vectors

$$\{ x^v / x \in X \}$$

in a Euclidean space

So for distinct $x, y \in X$

$$\langle x^v, y^v \rangle = - B_{xy} \leq 0$$

let v = θ_0-eigenvector for B.

Write

$$v = \sum_{x \in X} v_x \hat{x} \qquad v_x \in \mathbb{R}$$

We have

$$(\theta_0 I - B) v = 0,$$

$$v^t (\theta_0 I - B) v = 0$$

$$\|$$

$$\| \sum_{x \in X} v_x x^v \|^2$$

So

$$\sum_{x \in X} v_x x^v = 0 \qquad (*)$$

Define $X^+ = \{x \in X \,/\, v_x > 0\}$

Replacing v by $-v$ if nec, WLOG

$$X^+ \neq \phi$$

Show $X^+ = X$

Define $X^- = \{x \in X \,/\, v_x \leq 0\}$

By $(*)$,

$$\sum_{x \in X^+} v_x x^v = -\sum_{y \in X^-} v_y y^v$$

So

$$\underset{\substack{\text{IV} \\ 0}}{\Big\| \sum_{x \in X^+} v_x x^v \Big\|^2} = \Big\langle \sum_{x \in X^+} v_x x^v, -\sum_{y \in X^-} v_y y^v \Big\rangle$$

$$= \sum_{x \in X^+} \sum_{y \in X^-} \underset{\substack{\vee \\ 0}}{v_x} \, \underset{\substack{\text{IV} \\ 0}}{(-v_y)} \, \underset{\substack{\wedge \\ 0}}{\langle x^v, y^v \rangle}$$

$$\leq 0$$

Both sides 0 so

$$\sum_{x \in X^+} v_x x^v = 0$$

For $y \in X^-$,

$$0 = \left\langle \sum_{x \in X^+} \nu_x x^\vee, y^\vee \right\rangle$$

$$= \sum_{x \in X^+} \underset{0}{\underset{\vee}{\nu_x}} \underset{0}{\underset{\leq}{\langle x^\vee, y^\vee \rangle}}$$

So

$$\underbrace{\langle x^\vee, y^\vee \rangle}_{\substack{\| \\ -B_{xy}}} = 0 \qquad \forall x \in X^+$$

We have shown

$$B_{xy} = 0 \qquad \forall x \in X^+ \quad \forall y \in X^-$$

$$B : \quad \begin{array}{c|cc} & X^+ & X^- \\ \hline X^+ & * & 0 \\ X^- & 0 & * \end{array}$$

B is connected and $X^+ \neq \phi$ so $X^- = \phi$.

Therefore $X^+ = X$

(ii) Assume the v_0-eigenspace has dim ≥ 2.
Then it contains lin indep vectors u, v.

Pick $x \in X$.
$\exists$ nonzero linear combination of u, v that has
x-coordinate 0. This contradicts (i).

(iii) Since B is symmetric, its eigenspaces
are mutually orthogonal.
But any pair of Frobenius vectors are not
orthogonal.

(iv) By (i)

(v), (vi) First assume B is bipartite.

then $-\theta_0$ is the minimal eigenvalue of B by LEM 26.

Next assume that B is not bipartite.

Let $\theta =$ minimal eigval of B

show $\theta > -\theta_0$

Suppose $\theta \le -\theta_0$

Let $v =$ θ-eigenvector for B

By (iii), v is not Frobenius.

obs
B^2 is symmetric, all-entries ≥ 0, and irreducible (by LEM 25)

B^2 has max'l eigenvalue θ^2 and

$$B^2 v = \theta^2 v$$

Applying (i) to B^2, we find that v is Frobenius, contr.

So $\theta > -\theta_0$

We return our attention to a graph $\Gamma = (X, R)$ with adjacency matrix A and spectrum

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_0 & m_1 & \cdots & m_r \end{pmatrix}$$

COR 28 Assume Γ is connected. Then

(i) every θ_0-eigenvector for A is Frobenius

(ii) $m_0 = 1$

(iii) no other eigenspace of A has a Frobenius vector

(iv) $\theta_0 \geq 0$

(v) every eigenvalue of Γ is at least $-\theta_0$

(vi) Γ has eigenvalue $-\theta_0$ iff Γ is bipartite.

$\square$

LEM 29 For a graph $\Gamma = (X, R)$

TFAE

(i) Γ is a disjoint union of complete graphs

(ii) every eigenvalue of Γ is at least -1.

pf (i) → (ii) We saw each complete graph K_n has eigenvalues $n-1, -1$

(ii) → (i) Define a binary relation $\sim$ on X:

For $x, y \in X$

$x \sim y$ whenever $x = y$ or x, y are adjacent.

Show $\sim$ is an equivalence relation.

Obs $I + A$ is pos semi def.

View $I + A$ as the inner product matrix for some vectors $\{x^\vee \mid x \in X\}$ in a Euclidean space.

For $x, y \in X$

$$\langle x^\vee, y^\vee \rangle = \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{if } x \not\sim y \end{cases}$$

So for $x, y \in X$,

$$\|x^\vee - y^\vee\|^2 = \underbrace{\|x^\vee\|^2}_{1} - 2\langle x^\vee, y^\vee \rangle + \underbrace{\|y^\vee\|^2}_{1}$$

$$= \begin{cases} 0 \text{ if } x \sim y \\ 2 \text{ if } x \not\sim y \end{cases}$$

So

$$x^\vee = y^\vee \text{ iff } x \sim y.$$

Therefore $\sim$ is an equiv rel. $\square$

For our graph $\Gamma = (X, R)$ define

$$k_{max} = \max\{k(x) \mid x \in X\}$$
$$k_{min} = \min\{k(x) \mid x \in X\}$$

LEM 30 Assume Γ is connected. Then the max'l eigenvalue θ_0 satisfies

$$k_{min} \leq \theta_0 \leq k_{max}.$$

pf let v = θ_0-eigenvector for A

So v is Frobenius.

Write $$v = \sum_{x \in X} v_x \hat{x} \qquad v_x \in \mathbb{R}$$

WLOG $$v_x > 0 \qquad x \in X$$

We have $$Av = \theta_0 v$$

So $$\theta v_x = \sum_{y \in \Gamma(x)} v_y \qquad x \in X$$

$\exists \ x \in X$ s.t

$$v_x \le v_y \qquad \forall y \in X$$

obs

$$\theta_0 v_x = \sum_{y \in \Gamma(x)} v_y \ge \sum_{y \in \Gamma(x)} v_x = k(x)\, v_x \ge k_{min}\, v_x$$

$v_x > 0$ so

$$\theta_0 \ge k_{min}$$

$\exists \ x \in X$ s.t

$$v_x \ge v_y \qquad \forall y \in X$$

obs

$$\theta_0 v_x = \sum_{y \in \Gamma(x)} v_y \le \sum_{y \in \Gamma(x)} v_x = k(x)\, v_x \le k_{max}\, v_x$$

$v_x > 0$ so

$$\theta_0 \le k_{max}$$

$\square$