

Combinatorics 475

Lecture 6

Def

Given sets X, Y Given function $f: X \rightarrow Y$ (i) f is called 1-1 whenever: $\forall$ distinct $x, x' \in X$

$$f(x) \neq f(x')$$

(ii) f is called onto whenever: $\forall y \in Y \exists x \in X$ such that

$$f(x) = y$$

(iii) f is a bijection whenever f is 1-1 and onto.

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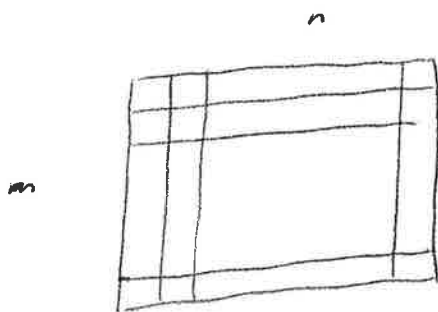
In terms of functions P P is saying the following:

(PPV2) Given finite sets X, Y Given a function $f: X \rightarrow Y$ If $|X| > |Y|$ then f is not 1-1.

Related statements:

- Assume $|X| = |Y|$ If f is onto then f is 1-1
- Assume $|X| = |Y|$ If f is 1-1 then f is onto

Ex

Given $m \times n$ chessboard

start at NW square and walk "SE" with wrap around

Will you eventually traverse every square?

Case $m=5$ $n=7$

X	Q	Q	X	Q	Q	X
Q	X	Q	Q	X	X	Q
X	Q	X	Q	Q	Q	Q
Q	Q	Q	X	Q	Q	X
Q	X	Q	Q	X	Q	Q

yes

Case $m=4$ $n=6$

Q		Q		Q	
	Q		Q		Q
X		Q		Q	
	Q		Q		Q

No

Call m, n relatively prime whenever they have no prime factor in common.

5, 7 rel. prime

4, 6 not rel. prime since 2 divides both

the Chinese remainder theorem says that you will traverse every square of the $m \times n$ chessboard whenever m, n are relatively prime.

we sharpen notation

Label the rows of the chessboard

$0, 1, \dots, m-1$

Label the cols of the chessboard

$0, 1, \dots, n-1$

For an integer $N \geq 1$ define the set

$$\mathbb{Z}_N = \{0, 1, \dots, N-1\}$$

View chessboard as Cartesian product

$$\mathbb{Z}_m \times \mathbb{Z}_n$$

Consider a function

$$f: \begin{array}{ccc} \mathbb{Z}_{mn} & \longrightarrow & \mathbb{Z}_m \times \mathbb{Z}_n \\ t & \longrightarrow & \text{Location on chessboard} \\ & & \text{after } t \text{ steps of the walk} \end{array}$$

obs:

f is onto $\Leftrightarrow$ the walk will traverse each square of the chessboard

$\forall t \in \mathbb{Z}_{mn}$ describe $f(t)$

write $f(t) = (a, b)$

meaning:

after t steps, located in row a , col b of chessboard

obs:

$a =$ remainder when divide t by m

$b =$ remainder when divide t by n

We now formally state Chinese Remainder Thm.

thm (Chinese Remainder thm)

Given rel prime pos integers m, n

Given $a \in \mathbb{Z}_m$ and $b \in \mathbb{Z}_n$

then $\exists x \in \mathbb{Z}_{mn}$ such that

$$x \equiv a \pmod{m} \quad \text{and} \quad x \equiv b \pmod{n}$$

pf Consider our function

$$f: \mathbb{Z}_{mn} \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_n$$

$$t \longrightarrow \left(\begin{array}{l} \text{remainder when divide} \\ t \text{ by } m \end{array}, \begin{array}{l} \text{remainder when} \\ \text{divide } t \text{ by } n \end{array} \right)$$

show f is onto.

$$\begin{aligned} \text{obs } |\mathbb{Z}_m \times \mathbb{Z}_n| &= |\mathbb{Z}_m| |\mathbb{Z}_n| \\ &= mn \\ &= |\mathbb{Z}_{mn}| \end{aligned}$$

By PP suf to show f is 1-1

Suppose f is not 1-1.

then $\exists x, x' \in \mathbb{Z}_{mn}$ ($x < x'$) such that

$$f(x) = f(x')$$

Call this common value (r, a)

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So

$$x \equiv r \pmod{m}$$

$$x \equiv a \pmod{n}$$

$$x' \equiv r \pmod{m}$$

$$x' \equiv a \pmod{n}$$

So

$$x' - x \equiv 0 \pmod{m}$$

$$x' - x \equiv 0 \pmod{n}$$

So

$$m \mid x' - x$$

$$n \mid x' - x$$

But m, n rel prime so

$$mn \mid x' - x$$

Now

$$x' - x \geq mn$$

But

$$0 \leq x < x' \leq mn - 1$$

So

$$x' - x \leq mn - 1$$

Cont.

Hence for 1-1.

Result follows.

□

Here is another application of PP

LEM Fix integer $n \geq 1$

Pick a subset

$$S \subseteq \{1, 2, \dots, 2n\}, \quad |S| = n+1$$

Then $\exists$ distinct $i, j \in S$ such that

$$i \mid j \text{ or } j \mid i$$

pf For an integer $x > 0$

factor x into primes:

$$x = 2^{n_1} \underbrace{3^{n_2} 5^{n_3} \dots}_{\text{odd part of } x}$$

Call this product x_{odd} , the
odd part of x

x_{odd} is the unique odd integer such that

$$x = x_{\text{odd}} \text{ times a power of } 2$$

Obs

$$1 \leq x_{\text{odd}} \leq x$$

x	1	2	3	4	5	6	7	8	9	10	...
x_{odd}	1	1	3	1	5	3	7	1	9	5	...

For integers $x, y > 0$ assume

$$x_{\text{odd}} = y_{\text{odd}}$$

Then x/y or y/x

Reasons: Write

$$x = m 2^r$$

$$y = m 2^s$$

$$(m = x_{\text{odd}} = y_{\text{odd}})$$

If $r \leq s$ then x/y

If $s \leq r$ then y/x

For $1 \leq i \leq 2n$ i_{odd} is contained in

$$\{1, 3, 5, \dots, 2n-1\}$$

(*)

Set (*) has cardinality n

Recall $|S| = n+r$, so by PP

$\exists$ distinct $i, j \in S$ such that

$$i_{\text{odd}} = j_{\text{odd}}$$

Now i/j or j/i

□

3.2 Pigeonhole principle: strong form

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We consider more versions of PP.

thm Given n boxes $\text{Box}_1, \dots, \text{Box}_n$

For $1 \leq i \leq n$ pick integer $q_i \geq 1$

We put

$$q_1 + q_2 + \dots + q_n - n + 1$$

objects in the boxes. Then:

Box 1 gets at least q_1 objects

or

Box 2 q_2 "

or

...

Box n q_n "

pf Suppose not. Then

Box 1 gets $\leq q_1 - 1$ objects

and Box 2 $\leq q_2 - 1$ "

and

...

and Box n $\leq q_n - 1$ objects

Total # objects must be $\leq$

$$(q_1 - 1) + (q_2 - 1) + \dots + (q_n - 1) = q_1 + q_2 + \dots + q_n - n$$

$$< q_1 + q_2 + \dots + q_n - n + 1, \text{ cont } \square$$

Another version of PP

Thm Given integers

$$m_1, m_2, \dots, m_n$$

($1 \leq n < \infty$)

Given integer q

Assume that the average value of $m_1, m_2, \dots, m_n$ is greater than $q-1$. Then $\exists i$ ($1 \leq i \leq n$) such that

$$m_i \geq q$$

pf Compute average:

$$\frac{m_1 + m_2 + \dots + m_n}{n} > q-1$$

(*)

Suppose the theorem is false. Then

$$m_i < q \quad (1 \leq i \leq n)$$

m_i, q are integers so

$$m_i \leq q-1 \quad (1 \leq i \leq n)$$

Now

$$\begin{aligned} \frac{m_1 + m_2 + \dots + m_n}{n} &\leq \frac{q-1 + q-1 + \dots + q-1}{n} \\ &= \frac{n(q-1)}{n} \\ &= q-1 \end{aligned}$$

Contradicts (*). So theorem is true.

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