Math 846

Lecture 6

Some asides about linear algebra

Until further notice

$$X = \text{nonempty finite set}$$

$$\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}$$

Given Hermitean $B \in \mathrm{Mat}_X(\mathbb{F})$

So B is diagonalizable, with all eigenvalues $\in \mathbb{R}$

Recall distinct eigenvalues $\{\theta_i\}_{i=0}^{r}$ for B.

From now on assume θ_0 is maximal, so

$$\theta_0 > \theta_i \qquad (1 \le i \le r)$$

Recall, B is positive definite whenever

$$\theta_i > 0 \qquad (0 \le i \le r)$$

B is positive semi definite whenever

$$\theta_i \geq 0 \qquad (0 \le i \le r)$$

Def 23 A <u>Hermitean space</u> is a finite dimensional vector space H over $\mathbb{F}$, together with a function

$$\langle , \rangle : H \times H \to \mathbb{F}$$

such that

(i) $\langle u, v \rangle = \overline{\langle v, u \rangle}$ $\quad u, v \in H$

(ii) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ $\quad u, v \in H \quad \alpha \in \mathbb{F}$

(iii) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ $\quad u, v, w \in H$

(iv) $\langle u, u \rangle \geq 0$ $\quad u \in H$

(v) $\langle u, u \rangle = 0$ iff $u = 0$ $\quad \forall u \in H$

Ex A Hermitean space $H, \langle , \rangle$ has a basis $\{v_i\}_{i=1}^n$ such that

$$\langle v_i, v_j \rangle = \delta_{ij} \quad 1 \leq i, j \leq n$$

"orthonormal basis"

Note A Hermitean space over $\mathbb{F}=\mathbb{R}$ is called a <u>Euclidean space</u>

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Prop 24 For $B \in \mathrm{Mat}_X(\mathbb{F})$ TFAE

(i) B is positive semi definite Hermitean

(ii) There exists a Hermitean space $H, \langle , \rangle$ and vectors $\{x^v \mid x \in X\}$ in H such that

$$B_{xy} = \langle x^v, y^v \rangle \quad \forall x, y \in X$$

Pf (i) → (ii) Since B is Hermitean $\exists$ diagonal $D \in \mathrm{Mat}_X(\mathbb{F})$ and invertible $P \in \mathrm{Mat}_X(\mathbb{F})$ st

$$B = P^{-1} D P \qquad P^{-1} = \bar{P}^t$$

Diagonal entries in D are the eigenvalues of B and hence nonneg real.

These entries have real square roots

$\exists$ diagonal matrix $\varphi \in \mathrm{Mat}_X(\mathbb{F})$ with real diagonal entries and $D = \varphi^2$

By construction

$$B = Y^t \bar{Y} \qquad Y = \overline{QP}$$

Define Hermitean space

$$H = \mathbb{F}^X$$

$$\langle u, v \rangle = u^t \bar{v} \qquad u, v \in H$$

For $x \in X$ define

$$x^\vee = \text{column } x \text{ of } Y$$
$$\in H$$

We have

$$B_{xy} = \langle x^\vee, y^\vee \rangle \qquad x, y \in X$$

(ii) $\to$ (i) For $x, y \in X$

$$B_{xy} = \langle x^\nu, y^\nu \rangle = \overline{\langle y^\nu, x^\nu \rangle} = \overline{B_{yx}}$$

so B is Hermitean.

show B is pos semi def.

For $v \in \mathbb{F}^X$, write

$$v = \sum_{x \in X} v_x \hat{x} \qquad v_x \in \mathbb{F}$$

obs

$$\bar{v}^t B v = \Big\| \sum_{x \in X} \bar{v}_x x^\nu \Big\|^2$$

$$\geq 0$$

For an eigvalue θ of B show $\theta \geq 0$

Let v = θ-eigenvector for B

$$Bv = \theta v$$

$$\underbrace{\bar{v}^t B v}_{\geq 0} = \theta \underbrace{\bar{v}^t v}_{> 0}$$

So $\theta \geq 0$

$\square$

Next we recall the Perron – Frobenius theorem for nonnegative matrices.

Given a symmetric $B \in \mathrm{Mat}_X(\mathbb{F})$

B is <u>reducible</u> whenever $\exists$ bipartition

$$X = X^+ \cup X^- \quad \text{(disj union of nonempty sets)}$$

such that

$$B_{xy} = 0 \qquad \forall x \in X^+ \quad \forall y \in X^-$$

$$B: \quad \begin{array}{c|cc} & X^+ & X^- \\ \hline X^+ & * & 0 \\ X^- & 0 & * \end{array}$$

B is <u>irreducible</u> iff B is not reducible

B is <u>bipartite</u> whenever $\exists$ bipartition

$$X = X^+ \cup X^- \quad \text{(disj union)}$$

such that for $x, y \in X$,

$$B_{xy} = 0 \quad \text{if} \quad x, y \in X^+ \quad \text{or} \quad x, y \in X^-$$

$$B: \quad \begin{array}{c|cc} & X^+ & X^- \\ \hline X^+ & 0 & * \\ X^- & * & 0 \end{array}$$

LEM 25 Given an irreducible symmetric $B \in Mat_X(\mathbb{F})$ Lec 6 8

(i) Assume B is bipartite and $|X| \geq 2$. Then B^2 is reducible.

(ii) Assume B^2 is reducible. Further assume that each entry of B is real and nonnegative. Then B is bipartite.

pf (i) Assume $B \neq 0$, else trivial.

$\exists$ bipartition

$$X = X^+ \cup X^- \qquad \text{(disj union)}$$

st for $x, y \in X$,

$$B_{xy} = 0 \quad \text{if} \quad x, y \in X^+ \text{ or } x, y \in X^-$$

Since $B \neq 0$,

$$X^+ \neq \phi, \qquad X^- \neq \phi.$$

For $x \in X^+$ and $y \in X^-$,

$$(B^2)_{xy} = \sum_{z \in X} \underbrace{\overset{=0 \text{ if } z \in X^+}{B_{xz}} \; \overset{=0 \text{ if } z \in X^-}{B_{zy}}}_{=0} = 0$$

So B^2 is reducible.

(ii) $\exists$ nonempty $X^+, X^- \subseteq X$ st

$$X = X^+ \cup X^- \quad \text{(disj union)}$$

and

$$(B^2)_{xy} = 0 \qquad x \in X^+ \quad y \in X^-$$

Show both

$$B_{xy} = 0 \qquad \forall x,y \in X^+ \qquad (*)$$

$$B_{xy} = 0 \qquad \forall x,y \in X^- \qquad (**)$$

To show $(*)$, define

$$X_+^+ = \{ x \in X^+ \mid \exists y \in X^+ \text{ s.t. } B_{xy} \neq 0 \}$$

Show $\quad X_+^+ = \phi$

<u>claim</u>

$$B_{xz} = 0 \qquad \forall x \in X_+^+ \qquad \forall z \in X \setminus X_+^+$$

<u>pf cl</u> Let x, z be given.

First assume $\quad z \in X^-$

Since $x \in X_+^+$,

$\exists\, y \in X^+$ st $B_{xy} \neq 0$

Since $y \in X^+$ and $z \in X^-$

$$(B^2)_{yz} = 0$$

We have

$$\underset{0}{\overset{\shortparallel}{(B^2)_{yz}}} = \sum_{w \in X} B_{yw} B_{wz}$$

$$= \underset{0}{\underset{\vee}{B_{yx}}}\ \underset{0}{\underset{\text{IV}}{B_{xz}}} + \sum_{w \in X \setminus x} \underset{0}{\underset{\text{IV}}{B_{yw}}}\ \underset{0}{\underset{\text{IV}}{B_{wz}}}$$

So $B_{xz} = 0$

Next assume $z \in X^+$

Since $z \notin X_+^+$,

$$B_{xz} = 0$$

claim proved ✓ We have

$$X \setminus X_+^+ \supseteq X^- \neq \phi$$

Now $X_+^+ = \phi$ since B is irreducible.

This shows $(*)$, and $(**)$ is similar $\square$

Here is a variation on LEM 3

LEM 26 Given a bipartite diagonalizable $B \in Mat_X(\mathbb{F})$.

If θ is an eigenvalue of B, then so is $-\theta$.

Moreover, the eigenspaces for θ and $-\theta$ have the same dimension.

pf $\exists$ bipartition

$$X = X^+ \cup X^- \quad (\text{disj union})$$

st $\forall x, y \in X$,

$$B_{xy} = 0 \quad \text{if } x, y \in X^+ \text{ or } x, y \in X^-$$

Define a diagonal matrix $\Delta \in Mat_X(\mathbb{F})$

s.t
$$\Delta_{xx} = \begin{cases} 1 & \text{if } x \in X^+ \\ -1, & \text{if } x \in X^- \end{cases} \qquad x \in X$$

obs Δ is invertible and

$$B\Delta = -\Delta B$$

Fn $v \in V$ and $\theta \in \mathbb{F}$,

$$Bv = \theta v \quad \text{iff} \quad B\Delta v = -\theta \Delta v$$

Result follows. $\square$