

Combinatorics 475

Lecture 5

Ch 3 The pigeonhole principle (PP)

3.1 PP: simple form

thm (PP) For an integer $n \geq 1$, if we put $n+1$ objects in n boxes, then some box gets at least 2 objects.

Ex. Among 13 people there are 2 that have birthdays in same month.

Ex Given 21 people consisting of n married couples. We pick $n+1$ people. Then among them are a married couple.

To motivate the next application of PP
we make some observations.

- For $a, b \in \mathbb{Z}$ at least one of
 $a, b, a+b$

is even

- For $a, b, c \in \mathbb{Z}$ at least one of
 $a, b, c, a+b, b+c, a+b+c$

is divisible by 3

- For $a, b, c, d \in \mathbb{Z}$ at least one of

$$a \quad b \quad c \quad d$$

$$a+b \quad b+c \quad c+d$$

$$a+b+c \quad b+c+d$$

$$a+b+c+d$$

is divisible by 4

In general we have the following.

LEM Given integers

$$a_1, a_2, \dots, a_m$$

$$(m \geq 1)$$

at least one of the sums

$$a_{i_1} + \dots + a_{i_{j-1}} + a_j$$

$$0 \leq i < j \leq m$$

is divisible by m .

pf For $x, y \in \mathbb{Z}$ write

$$x \equiv y \pmod{m}$$

"x equals y modulo m"

whenever m divides $x - y$.

obs: $\forall x \in \mathbb{Z} \exists$ unique $r \in \{0, 1, 2, \dots, m-1\}$

such that

$$x \equiv r \pmod{m}$$

Indeed r is remainder when we divide x by m .

Consider $m+1$ sums

$$0$$

$$a_1$$

$$a_1 + a_2$$

$$a_1 + a_2 + a_3$$

$$\vdots$$

$$a_1 + a_2 + \dots + a_m$$

Each sum is equal to exactly one of $\{0, 1, 2, \dots, m-1\}$
mod m .

By PP $\exists i, j$ ($0 \leq i < j \leq m$) such that

$$a_1 + a_2 + \dots + a_i \equiv a_1 + a_2 + \dots + a_j \pmod{m}$$

Now

$$\ominus \equiv \underbrace{a_1 + a_2 + \dots + a_j - (a_1 + a_2 + \dots + a_i)}_{a_{i+1} + \dots + a_{j-1} + a_j} \pmod{m}$$

So

m divides $a_{i+1} + \dots + a_{j-1} + a_j$

□

Given an integer $n \geq 1$

Consider a permutation $a_1 a_2 \dots a_n$ of $\{1, 2, \dots, n\}$

ex $n = 5$

2 5 4 1 3

Consider "decreasing subsequences" and "increasing subsequences"

r	decreasing subsequences length r					
3	541	543				
2	21	54	51	53	41	43
1	2	5	4	1	3	

r	increasing subsequences length r				
2	25	24	23	13	
1	2	5	4	1	3

For a sequence $a_1 a_2 \dots a_n$ by a subsequence of length r

we mean $a_{i_1} a_{i_2} \dots a_{i_r}$ where $1 \leq i_1 < i_2 < \dots < i_r \leq n$

this subsequence is decreasing if $a_{i_1} > a_{i_2} > \dots > a_{i_r}$

or increasing if $a_{i_1} < a_{i_2} < \dots < a_{i_r}$

Thm Given integer $r \geq 0$

$$\text{let } n = r^2 + 1$$

Then Each perm of $\{1, 2, \dots, n\}$ has either

(i) at least one increasing subsequence of length $r+1$

or

(ii) at least one decreasing subsequence of length $r+1$

Pf. We assume (i) fails and obtain (ii)

Call the permutation $a_1, a_2, \dots, a_n$

For $1 \leq i \leq n$ define

$l_i = \text{max'l length of an increasing subsequence of } a_1, a_2, \dots, a_n \text{ that ends at } a_i$

ex $r=3$ $n=10$

a_i	3	10	6	2	5	9	1	8	4	7
l_i	1	2	2	1	2	3	1	3	2	3

By assumption

$$l_i \leq r \quad 1 \leq i \leq n$$

Note: for $1 \leq i < j \leq n$ such that $a_i < a_j$ we have

$$l_i < l_j$$

Since for any increasing subsequence of length l_i that ends at a_i we could append a_j to get an increasing subsequence of length $l_i + 1$ that ends at a_j .

For $1 \leq l \leq r$ define

$$S_l = \{ i \mid 1 \leq i \leq n, a_i = l \}$$

By the note, the elements of S_l ^(induced) form a decreasing subsequence

ex

l	decreasing subsequence induced by S_l
1	3 2 1
2	10 6 5 4
3	9 8 7

$S_1, S_2, \dots, S_r$ partition $\{1, 2, \dots, n\}$

$$\text{So } \sum_{l=1}^r |S_l| = n = r^2 + 1$$

By PP $\exists l$ ($1 \leq l \leq r$) such that

$$|S_l| \geq r+1$$

Therefore $\exists$ decreasing subsequence of length $\geq r+1$. $\square$

LEM A student has 37 days to prepare for an exam.

- On each day $1, 2, \dots, 37$ she studies a certain number of hours (varies with day).
- She requires at most 60 study hours in total.
- She studies at least 1 hour each day.

Then $\exists$ a sequence of consecutive days during which she studies exactly 13 hours.

Pf For $1 \leq i \leq 37$ let

$b_i = \#$ hours she studies on day i

So

$$b_i \geq 1$$

Consider the numbers

$$\left\{ b_1 + b_2 + \dots + b_7 + 13 \right\}_{i=0}^{36} \cup \left\{ b_1 + b_2 + \dots + b_7 \right\}_{j=1}^{37}$$

(*)

We have $37 \times 2 = 74$ numbers

All among

$1, 2, \dots, 72$

By P.P numbers (*) are not distinct

So $\exists i, j$ ($0 \leq i < j \leq 37$) such that

$$b_1 + b_2 + \dots + b_i + 13 = b_1 + b_2 + \dots + b_j$$

Now

$$13 = b_{i+1} + \dots + b_j$$

During the days $i+1, \dots, j$ she studies exactly 13 hours $\square$

